# The spectral shift function for Dirac operators with electrostatic $\delta$-shell interactions

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# Contents

# PART I

## Spectral shift function

# A very brief history of the spectral shift function

This introduction is based on:

- BirmanYafaev'92: The spectral shift function. The papers of M. G. Krein and their further development
- BirmanPushnitski'98: Spectral shift function, amazing and multifaceted
- Yafaev'92 and '10: Mathematical Scattering Theory

**General assumption**

$\mathcal{H}$ Hilbert space, $A, B$ selfadjoint (unbounded) operators in $\mathcal{H}$

**I. M. Lifshitz, 1952**

$B - A$ **finite rank** operator. Then exists $\xi : \mathbb{R} \to \mathbb{R}$ such that *formally*

$$\operatorname{tr}\left(\varphi(B) - \varphi(A)\right) = \int_{\mathbb{R}} \varphi'(t)\xi(t)\,dt$$

# Krein's spectral shift function (1953 and 1962)

**Theorem**

Assume $B - A$ **trace class** operator, i.e. $B - A \in \mathfrak{S}_1$. Then exists real-valued $\xi \in L^1(\mathbb{R})$ such that

$$\mathrm{tr}\,\big((B-\lambda)^{-1} - (A-\lambda)^{-1}\big) = -\int_{\mathbb{R}} \frac{\xi(t)}{(t-\lambda)^2}\,dt$$

and $\int_{\mathbb{R}} \xi(t)\,dt = \mathrm{tr}\,(B-A)$.

- $\mathrm{tr}\,\big(\varphi(B) - \varphi(A)\big) = \int_{\mathbb{R}} \varphi'(t)\xi(t)\,dt$ for $\varphi(t) = (t-\lambda)^{-1}$
- Extends to Wiener class $W_1(\mathbb{R})$: $\varphi'(t) = \int e^{-it\mu}\,d\sigma(\mu)$

**Corollary**

If $\delta = (a,b)$ and $\overline{\delta} \cap \sigma_{\mathrm{ess}}(A) = \emptyset$ then

$$\xi(b-) - \xi(a+) = \dim\operatorname{ran} E_B(\delta) - \dim\operatorname{ran} E_A(\delta)$$

- Spectral shift function for $U, V$ unitary, $V - U \in \mathfrak{S}_1$

# Krein's spectral shift function (1953 and 1962)

## Theorem

Assume

$$(B-\lambda)^{-1}-(A-\lambda)^{-1}\in\mathfrak{S}_1,\qquad \lambda\in\rho(A)\cap\rho(B). \qquad (1)$$

Then exists $\xi\in L^1_{\rm loc}(\mathbb{R})$ such that $\int_{\mathbb{R}}|\xi(t)|(1+t^2)^{-1}\,dt<\infty$ and

$$\operatorname{tr}\big((B-\lambda)^{-1}-(A-\lambda)^{-1}\big)=-\int_{\mathbb{R}}\frac{\xi(t)}{(t-\lambda)^2}\,dt.$$

The function $\xi$ is unique up to a real constant.

- Trace formula for $\varphi(t)=(t-\lambda)^{-1}$ and $\varphi(t)=(t-\lambda)^{-k}$
- Large class of $\varphi$ in trace formula in Peller'85

## Birman-Krein formula

Assume (1). The scattering matrix $\{S(\lambda)\}$ of $\{A,B\}$ satisfies

$$\det S(\lambda)=e^{-2\pi i\xi(\lambda)}\qquad\text{for a.e. } \lambda\in\mathbb{R}$$

# Krein's spectral shift function: Generalizations

## L.S. Koplienko 1971

Assume $\rho(A) \cap \rho(B) \cap \mathbb{R} \neq \emptyset$ and for some $m \in \mathbb{N}$:

$$(B-\lambda)^{-m} - (A-\lambda)^{-m} \in \mathfrak{S}_1. \tag{2}$$

Then exists $\xi \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that $\int_{\mathbb{R}} |\xi(t)|(1+|t|)^{-(m+1)}\,dt < \infty$

$$\operatorname{tr}\left((B-\lambda)^{-m} - (A-\lambda)^{-m}\right) = \int_{\mathbb{R}} \frac{-m}{(t-\lambda)^{m+1}}\,\xi(t)\,dt.$$

## D.R. Yafaev 2005

Assume (2) for some $m \in \mathbb{N}$ odd. Then exists $\xi \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that $\int_{\mathbb{R}} |\xi(t)|(1+|t|)^{-(m+1)}\,dt < \infty$

$$\operatorname{tr}\left((B-\lambda)^{-m} - (A-\lambda)^{-m}\right) = \int_{\mathbb{R}} \frac{-m}{(t-\lambda)^{m+1}}\,\xi(t)\,dt.$$

# Some references and contributors

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# PART II

## Representation of the SSF via Weyl function

# Quasi boundary triples

$S \subset S^*$ closed symmetric operator in $\mathcal{H}$ with infinite defect

**Def. [Bruk76, Kochubei75; DerkachMalamud95; B. Langer07]**

$\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ **quasi boundary triple for $S^*$** if $\mathcal{G}$ Hilbert space and $T \subset \overline{T} = S^*$ and $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G}$ such that

(i) $(Tf, g) - (f, Tg) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g)$, $f, g \in \operatorname{dom} T$

(ii) $\Gamma := \binom{\Gamma_0}{\Gamma_1} : \operatorname{dom} T \to \mathcal{G} \times \mathcal{G}$ dense range

(iii) $A = T \upharpoonright \ker \Gamma_0$ selfadjoint

**Example 1: ($-\Delta + V$ on domain $\Omega$, $\partial\Omega$ of class $C^2$, $V \in L^\infty$ real)**

$$Sf = -\Delta f + Vf \upharpoonright \{f \in H^2(\Omega) : f|_{\partial\Omega} = \partial_\nu f|_{\partial\Omega} = 0\}$$
$$S^* f = -\Delta f + Vf \upharpoonright \{f \in L^2(\Omega) : \Delta f \in L^2(\Omega)\} \not\subset H^s(\Omega), \quad s > 0$$
$$Tf = -\Delta f + Vf \upharpoonright H^2(\Omega)$$

Here $(Tf, g) - (f, Tg) = (f|_{\partial\Omega}, \partial_\nu g|_{\partial\Omega}) - (\partial_\nu f|_{\partial\Omega}, g|_{\partial\Omega})$.
Choose $\mathcal{G} = L^2(\partial\Omega)$, $\Gamma_0 f := \partial_\nu f|_{\partial\Omega}$, $\Gamma_1 f := f|_{\partial\Omega}$.

# Example 2: Dirac operators with $\delta$-shell interactions

$\Sigma$ boundary of bdd. $C^\infty$-domain $\Omega_+ \subset \mathbb{R}^3$, $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}_+$

$$Sf = \begin{pmatrix} -ic\nabla f_+ + mc^2\beta f_+ \\ -ic\nabla f_- + mc^2\beta f_- \end{pmatrix}, \quad \operatorname{dom} S = H^1_0(\Omega_+) \oplus H^1_0(\Omega_-)$$

$$Tf = \begin{pmatrix} -ic\nabla f_+ + mc^2\beta f_+ \\ -ic\nabla f_- + mc^2\beta f_- \end{pmatrix}, \quad \operatorname{dom} T = H^1(\Omega_+) \oplus H^1(\Omega_-)$$

$$S^*f = \begin{pmatrix} -ic\nabla f_+ + mc^2\beta f_+ \\ -ic\nabla f_- + mc^2\beta f_- \end{pmatrix}, \quad \operatorname{dom} S^* \text{ maximal in } L^2$$

For $f \in \operatorname{dom} T = H^1(\Omega_+) \oplus H^1(\Omega_-)$ set

$$\Gamma_0 f := ic\alpha\cdot\nu(f_+|_\Sigma - f_-|_\Sigma) \quad \text{and} \quad \Gamma_1 f := \frac{1}{2}(f_+|_\Sigma + f_-|_\Sigma) + \frac{1}{\eta}\Gamma_0 f.$$

# Example 2: Dirac operators with $\delta$-shell interactions

**Proposition**

Then $\overline{T} = S^*$ and $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$ QBT, where

$$\Gamma_0 f := ic\alpha \cdot \nu(f_+|_\Sigma - f_-|_\Sigma) \quad \text{and} \quad \Gamma_1 f := \frac{1}{2}(f_+|_\Sigma + f_-|_\Sigma) + \frac{1}{\eta}\Gamma_0 f$$

for $f \in \operatorname{dom} T = H^1(\Omega_+) \oplus H^1(\Omega_-)$.

With this choice of $\Gamma_0$ and $\Gamma_1$ we have

$$A = T \upharpoonright \ker \Gamma_0 = -ic\nabla + mc^2\beta, \qquad \operatorname{dom} A = H^1(\mathbb{R}^3),$$

and

$$A_\eta := T \upharpoonright \ker \Gamma_1 = -ic\nabla + mc^2\beta + \eta\delta_\Sigma$$

$A_\eta$ selfadjoint in $L^2(\mathbb{R}^3)$ for $\eta \neq \{\pm 2c, 0\}$ (Holzmann lecture!)

# $\gamma$-field and Weyl function

$S \subset T \subset \overline{T} = S^*$, $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ quasi boundary triple (QBT).

**Definition**

Let $f_\lambda \in \ker(T - \lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$. $\gamma$-**field** and **Weyl function**:

$$\gamma(\lambda) : \mathcal{G} \to \mathcal{H}, \qquad \Gamma_0 f_\lambda \mapsto f_\lambda, \qquad \lambda \in \rho(A)$$
$$M(\lambda) : \mathcal{G} \to \mathcal{G}, \qquad \Gamma_0 f_\lambda \mapsto \Gamma_1 f_\lambda, \qquad \lambda \in \rho(A)$$

- $\gamma(\lambda)$ solves boundary value problem in PDE
- $M(\lambda)$ Dirichlet-to-Neumann (Neumann-to-Dir. map) in PDE

**Example 1: $-\Delta + V$, QBT $\{L^2(\partial\Omega), \partial_\nu f|_{\partial\Omega}, f|_{\partial\Omega}\}$**

Here $\ker(T - \lambda) = \{f \in H^2(\Omega) : -\Delta f + Vf = \lambda f\}$ and

$$\gamma(\lambda) : L^2(\partial\Omega) \supset H^{1/2}(\partial\Omega) \to L^2(\Omega), \quad \varphi \mapsto f_\lambda,$$

where $(-\Delta + V)f_\lambda = \lambda f_\lambda$ and $\partial_\nu f_\lambda|_{\partial\Omega} = \varphi$, and

$$M(\lambda) : L^2(\partial\Omega) \supset H^{1/2}(\partial\Omega) \to L^2(\partial\Omega), \quad \varphi = \partial_\nu f_\lambda|_{\partial\Omega} \mapsto f_\lambda|_{\partial\Omega}.$$

# Example 2: Dirac operators with $\delta$-shell interactions

$$Tf = \begin{pmatrix} -ic\nabla f_+ + mc^2\beta f_+ \\ -ic\nabla f_- + mc^2\beta f_- \end{pmatrix}, \quad \operatorname{dom} T = H^1(\Omega_+) \oplus H^1(\Omega_-)$$

and QBT $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0 f = ic\alpha \cdot \nu(f_+|_\Sigma - f_-|_\Sigma), \qquad \Gamma_1 f = \frac{1}{2}(f_+|_\Sigma + f_-|_\Sigma) + \frac{1}{\eta}\Gamma_0 f,$$

and $((A-\lambda)^{-1}f)(x) = \int_{\mathbb{R}^3} G_\lambda(x-y)f(y)dy$.

**$\gamma$-field and Weyl function defined on $\operatorname{ran} \Gamma_0 = H^{1/2}(\Sigma)$ are**

$$\gamma(\lambda) : L^2(\Sigma) \to L^2(\mathbb{R}^3), \quad \varphi \mapsto \int_\Sigma G_\lambda(x-y)\varphi(y)d\sigma(y),$$

$$M(\lambda) : L^2(\Sigma) \to L^2(\Sigma), \quad \varphi \mapsto \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} G_\lambda(x-y)\varphi(y)d\sigma(y) + \frac{1}{\eta}\varphi$$

# Quasi boundary triples and selfadjoint extensions

Perturbation problems for selfadjoint operators in QBT scheme:

**Lemma**

Assume $A, B$ selfadjoint and $S = A \cap B$,

$$Sf := Af = Bf, \quad \operatorname{dom} S = \{ f \in \operatorname{dom} A \cap \operatorname{dom} B : Af = Bf \}$$

*densely defined*. Then exists $T \subset \overline{T} = S^*$ and QBT $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ such that

$$A = T \upharpoonright \ker \Gamma_0 \quad \text{and} \quad B = T \upharpoonright \ker \Gamma_1$$

and

$$(B - \lambda)^{-1} - (A - \lambda)^{-1} = -\gamma(\lambda) M(\lambda)^{-1} \gamma(\bar{\lambda})^*,$$

where $\gamma$ and $M$ are the $\gamma$-field and Weyl function of $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$.

# Main abstract result: First order case

**Theorem**

$A, B$ selfadjoint, $S = A \cap B$ densely defined and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ QBT

$$A = T \upharpoonright \ker \Gamma_0 \quad \text{and} \quad B = T \upharpoonright \ker \Gamma_1$$

Assume

$$(A - \mu)^{-1} \geq (B - \mu)^{-1} \quad \text{for some } \mu \in \rho(A) \cap \rho(B) \cap \mathbb{R}$$

and $\overline{\gamma(\lambda_0)} \in \mathfrak{S}_2$, $M(\lambda_1)^{-1}, M(\lambda_2)$ bounded for $\lambda_0, \lambda_1, \lambda_2$ Then:

- $(B - \lambda)^{-1} - (A - \lambda)^{-1} = -\gamma(\lambda) M(\lambda)^{-1} \gamma(\bar{\lambda})^* \in \mathfrak{S}_1$
- $\operatorname{Im} \log(\overline{M(\lambda)}) \in \mathfrak{S}_1(\mathcal{G})$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and

$$\xi(t) = \lim_{\varepsilon \to +0} \frac{1}{\pi} \operatorname{tr}\left(\operatorname{Im} \log(\overline{M(t + i\varepsilon)})\right) \quad \text{for a.e. } t \in \mathbb{R}$$

is a spectral shift function for $\{A, B\}$, in particular,

$$\operatorname{tr}\left((B - \lambda)^{-1} - (A - \lambda)^{-1}\right) = -\int_{\mathbb{R}} \frac{1}{(t - \lambda)^2}\, \xi(t)\, dt.$$

# Main abstract result: Higher order case

**Theorem**

$A, B$ selfadjoint, $S = A \cap B$ densely defined and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ QBT

$$A = T \upharpoonright \ker \Gamma_0 \quad \text{and} \quad B = T \upharpoonright \ker \Gamma_1$$

Assume

$$(A - \mu)^{-1} \geq (B - \mu)^{-1} \quad \text{for some } \mu \in \rho(A) \cap \rho(B) \cap \mathbb{R}$$

$M(\lambda_1)^{-1}, M(\lambda_2)$ bounded for $\lambda_1, \lambda_2$ and for some $k \in \mathbb{N}$:

$$\frac{d^p}{d\lambda^p}\overline{\gamma(\lambda)}\frac{d^q}{d\lambda^q}\left(M(\lambda)^{-1}\gamma(\bar{\lambda})^*\right) \in \mathfrak{S}_1, \quad p + q = 2k,$$

$$\frac{d^q}{d\lambda^q}\left(M(\lambda)^{-1}\gamma(\bar{\lambda})^*\right)\frac{d^p}{d\lambda^p}\overline{\gamma(\lambda)} \in \mathfrak{S}_1, \quad p + q = 2k,$$

$$\frac{d^j}{d\lambda^j}\overline{M(\lambda)} \in \mathfrak{S}_{\frac{2k+1}{j}}, \qquad j = 1, \ldots, 2k + 1.$$

# Main abstract result: Higher order case

**Theorem**

$A, B$ selfadjoint, $S = A \cap B$ densely defined and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ QBT

Assume
$$A = T \upharpoonright \ker \Gamma_0 \quad \text{and} \quad B = T \upharpoonright \ker \Gamma_1$$

$$(A-\mu)^{-1} \geq (B-\mu)^{-1} \quad \text{for some } \mu \in \rho(A) \cap \rho(B) \cap \mathbb{R}$$

$M(\lambda_1)^{-1}, M(\lambda_2)$ bounded for $\lambda_1, \lambda_2$ and $\mathfrak{S}_p$-conditions. Then:

- $(B-\lambda)^{-(2k+1)} - (A-\lambda)^{-(2k+1)} \in \mathfrak{S}_1$
- For any ONB $(\varphi_j)$ in $\mathcal{G}$ and a.e. $t \in \mathbb{R}$

$$\xi(t) = \sum_j \lim_{\varepsilon \to +0} \frac{1}{\pi} \big(\operatorname{Im} \log(\overline{M(t+i\varepsilon)})\varphi_j, \varphi_j\big)$$

is a spectral shift function for $\{A, B\}$, in particular,

$$\operatorname{tr}\left((B-\lambda)^{-(2k+1)} - (A-\lambda)^{-(2k+1)}\right) = -\int_{\mathbb{R}} \frac{2k+1}{(t-\lambda)^{2k+2}}\, \xi(t)\, dt$$

# Remarks

- If $A, B$ semibounded then

$$(A-\mu)^{-1} \geq (B-\mu)^{-1} \quad \Longleftrightarrow \quad A \leq B$$

in accordance with $\xi(t) = \frac{1}{\pi}\mathrm{tr}\,(\mathrm{Im}\log(\overline{M(t+i0)})) \geq 0$

Representation of SSF via $M$-function:

- Rank 1, $k = 0$ [LangerSnooYavrian'01]
- Rank $n < \infty$, $k = 0$ [B. MalamudNeidhardt'08]
- Other representation via modified perturbation determinant for $M$ for $k = 0$ [MalamudNeidhardt'15]

Representation of scattering matrix $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$ of $\{A, B\}$ for the trace class case ($k = 0$):

$$S(\lambda) = I - 2i\sqrt{\mathrm{Im}\, M(\lambda+i0)}\left(\overline{M(\lambda+i0)}\right)^{-1}\sqrt{\mathrm{Im}\, M(\lambda+i0)}$$

[AdamyanPavlov'86], [B. MalamudNeidhardt'08 and '15], [MantilePosilicanoSini'15]

# PART III

## Dirac operators with electrostatic $\delta$-shell interactions

# A QBT for Dirac operators

$\Sigma$ boundary of bdd. $C^\infty$-domain $\Omega_+ \subset \mathbb{R}^3$, $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}_+$

$$Sf = \begin{pmatrix} -ic\nabla f_+ + mc^2\beta f_+ \\ -ic\nabla f_- + mc^2\beta f_- \end{pmatrix}, \quad \mathrm{dom}\, S = H_0^1(\Omega_+) \oplus H_0^1(\Omega_-)$$

$$Tf = \begin{pmatrix} -ic\nabla f_+ + mc^2\beta f_+ \\ -ic\nabla f_- + mc^2\beta f_- \end{pmatrix}, \quad \mathrm{dom}\, T = H^1(\Omega_+) \oplus H^1(\Omega_-)$$

Then $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$ QBT, where

$$\Gamma_0 f := ic\alpha \cdot \nu(f_+|_\Sigma - f_-|_\Sigma) \quad \text{and} \quad \Gamma_1 f := \frac{1}{2}(f_+|_\Sigma + f_-|_\Sigma) + \frac{1}{\eta}\Gamma_0 f$$

and $A = T \upharpoonright \ker \Gamma_0 = -ic\nabla + mc^2\beta$ free Dirac operator,

$$A_\eta = B = T \upharpoonright \ker \Gamma_1 = -ic\nabla + mc^2\beta + \eta\delta_\Sigma$$

# $\gamma$-field and Weyl function, strength $\eta$

$$\gamma(\lambda) : L^2(\Sigma) \to L^2(\mathbb{R}^3), \quad \varphi \mapsto \int_\Sigma G_\lambda(x-y)\varphi(y)d\sigma(y),$$

$$M(\lambda) : L^2(\Sigma) \to L^2(\Sigma), \quad \varphi \mapsto \lim_{\substack{\varepsilon\to 0 \\ |x-y|>\varepsilon}} \int G_\lambda(x-y)\varphi(y)d\sigma(y) + \frac{1}{\eta}\varphi$$

are continuous and admit closures defined on $L^2(\Sigma)$. Recall from [ArrizabalagaMasVega'15] that

$$\sup_{\lambda\in[-mc^2, mc^2]} \|\overline{M(\lambda)} - \eta^{-1}\| = M_0 < \infty.$$

## Assumptions on $\eta$

$$|\eta| \leq \frac{1}{M_0} \quad \text{and} \quad \eta \neq \{\pm 2c, 0\}.$$

In this case gap preserved: $\sigma(A_\eta) \cap (-mc^2, mc^2) = \emptyset$

# Spectral shift function for the pair $\{A, A_\eta\}$, $\eta \neq \pm 2c$

**Theorem**

- $(A_\eta - \lambda)^{-3} - (A - \lambda)^{-3} \in \mathfrak{S}_1(L^2(\mathbb{R}^3))$ for $\lambda \in \rho(A_0) \cap \rho(A_\eta)$
- For $\eta \in (0, M_0^{-1})$, any ONB $(\varphi_j)$ in $L^2(\Sigma)$ and a.e. $t \in \mathbb{R}$

$$\xi_+(t) = \sum \lim_{\varepsilon \to +0} \frac{1}{\pi} \left(\operatorname{Im} \log(\overline{M(t + i\varepsilon)})\varphi_j, \varphi_j\right)$$

  is a spectral shift function for $\{A, A_\eta\}$
- For $\eta \in (-M_0^{-1}, 0)$, any ONB $(\varphi_j)$ in $L^2(\Sigma)$ and a.e. $t \in \mathbb{R}$

$$\xi_-(t) = \sum \lim_{\varepsilon \to +0} \frac{1}{\pi} \left(\operatorname{Im} \log(\overline{-M(t + i\varepsilon)^{-1}})\varphi_j, \varphi_j\right)$$

  is a spectral shift function for $\{A_\eta, A\}$
- The following trace formula holds for $\lambda \in \rho(A_0) \cap \rho(A_\eta)$:

$$\operatorname{tr}\left((A_\eta - \lambda)^{-3} - (A - \lambda)^{-3}\right) = \mp \int_{\mathbb{R}} \frac{3}{(t - \lambda)^4} \, \xi_\pm(t) \, dt$$

# Sketch of the proof

- $\eta \in (0, M_0^{-1})$ and $\lambda \in (-mc^2, mc^2)$

$$
\begin{aligned}
(A_\eta - \lambda)^{-1} &= (A - \lambda)^{-1} - \gamma(\lambda)\big(\eta^{-1} + M(\lambda)\big)^{-1}\gamma(\lambda)^* \\
&\leq (A - \lambda)^{-1}
\end{aligned}
$$

- $\frac{d^k}{d\lambda^k}\gamma(\bar\lambda)^* = k!\Gamma_1(A-\lambda)^{-k-1} \in \mathfrak{S}_{\frac{4}{2k+1}}\big(L^2(\mathbb{R}^3), L^2(\Sigma)\big)$
- $\frac{d^k}{d\lambda^k}\overline{\gamma(\lambda)} \in \mathfrak{S}_{\frac{4}{2k+1}}\big(L^2(\Sigma), L^2(\mathbb{R}^3)\big)$
- $\frac{d^k}{d\lambda^k}\overline{M(\lambda)} = k!\Gamma_1(A-\lambda)^{-k}\gamma(\lambda) \in \mathfrak{S}_{2/k}(L^2(\Sigma))$

Use $\mathfrak{S}_{1/x} \cdot \mathfrak{S}_{1/y} = \mathfrak{S}_{1/(x+y)}$ and conclude

- $\frac{d^p}{d\lambda^p}\overline{\gamma(\lambda)}\frac{d^q}{d\lambda^q}\big(M(\lambda)^{-1}\gamma(\bar\lambda)^*\big) \in \mathfrak{S}_1, \quad p+q=2,$
- $\frac{d^q}{d\lambda^q}\big(M(\lambda)^{-1}\gamma(\bar\lambda)^*\big)\frac{d^p}{d\lambda^p}\overline{\gamma(\lambda)} \in \mathfrak{S}_1, \quad p+q=2,$
- $\frac{d^j}{d\lambda^j}\overline{M(\lambda)} \in \mathfrak{S}_{3/j}, \qquad j=1,2,3.$

Apply Main Theorem with $k=1$.

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# Thank you for your attention