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Large time asymptotics for the Dirac equation

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Linear and Nonlinear Dirac Equation: Advances and Open Problems

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Linear Dirac equation

$$\left.\begin{aligned} -i\gamma^\mu\partial_\mu\psi + M\psi &= 0 \\ \psi(0) &= f \end{aligned}\right\} \quad \text{on } (t,x)\in\mathbb{R}\times\mathbb{R}^n.$$

- the spinor $\psi(t,x) : \mathbb{R}^{1+n} \to \mathbb{C}^N$ and mass $M \geqslant 0$.
- Repeated Greek indices are summed over $\mu = 0, ..., n$, and $\partial_0 = \partial_t$, $\partial_j = \partial_{x_j}$ ($j \geqslant 1$), so

$$\gamma^\mu\partial_\mu = \gamma^0\partial_t + \sum_{j=1}^{3}\gamma^j\partial_j.$$

- γ^μ are (constant) $N\times N$ complex matrices such that

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I_{N\times N}, \qquad (\gamma^0)^\dagger = \gamma^0, \qquad (\gamma^j)^\dagger = -\gamma^j$$

  and $g^{\mu\nu} = \text{diag}(1,-1,-1,-1)$. In particular,

$$(-i\gamma^\mu\partial_\mu + M)^\dagger(-i\gamma^\mu\partial_\mu + M) = \partial_t^2 - \Delta + M^2 = \Box + M^2.$$

Linear Dirac equation

$$\left.\begin{aligned} -i\gamma^\mu \partial_\mu \psi + M\psi &= 0 \\ \psi(0) &= f \end{aligned}\right\} \quad \text{on } (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

- If $n = 3$, one choice is

$$\gamma^0 = \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

  where the Pauli matrices σ^j are defined as

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- If $n = 1, 2$, we take

$$\gamma^0 = \sigma^3, \qquad \gamma^1 = i\sigma^2, \qquad \gamma^2 = -i\sigma^1.$$

Basic Estimates

$$\left.\begin{aligned} -i\gamma^\mu \partial_\mu \psi + M\psi = 0 \\ \psi(0) = f \end{aligned}\right\} \quad \text{on } (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

- Energy Estimate

$$\|\psi\|_{L^\infty_t H^s_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|f\|_{H^s(\mathbb{R}^n)} + \|(-i\gamma^\mu\partial_\mu + M)\psi\|_{L^1_t H^s_x(\mathbb{R}^{1+n})}$$

- Dispersive Bound

$$\|\psi(t)\|_{L^\infty_x(\mathbb{R}^n)} \lesssim t^{-\frac{n-1}{2}} \sum_{|\kappa|\leqslant n+1} \|\nabla^\kappa f\|_{L^1_x(\mathbb{R}^n)}$$

Basic Estimates

$$\left.\begin{aligned} -i\gamma^\mu \partial_\mu \psi + M\psi = 0 \\ \psi(0) = f \end{aligned}\right\} \text{ on } (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

- L^∞ Strichartz

  Let $\frac{1}{q} < \min\{\frac{n-1}{4}, \frac{1}{2}\}$. Then

$$\|\psi\|_{L^q_t L^\infty_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|f\|_{H^{\frac{n}{2}-\frac{1}{q}}(\mathbb{R}^n)} + \|(-i\gamma^\mu\partial_\mu + M)\psi\|_{L^1_t H_x^{\frac{n}{2}-\frac{1}{q}}(\mathbb{R}^{1+n})}$$

(see [Strichartz'97],[Ginibre-Velo'89],[Escobedo-Vega '97]...). Key points:

1. Decay in time,
2. Saves $\frac{1}{q}$ derivatives over Sobolev embedding

$$\begin{aligned} \|\psi(t)\|_{L^\infty_x(\mathbb{R}^n)} &\lesssim \|\psi(t)\|_{H^{\frac{n}{2}+\epsilon}(\mathbb{R}^n)} \\ &\lesssim \|f\|_{H^{\frac{n}{2}+\epsilon}(\mathbb{R}^n)} + \|(-i\gamma^\mu\partial_\mu + M)\psi\|_{L^1_t H_x^{\frac{n}{2}+\epsilon}(\mathbb{R}^{1+n})} \end{aligned}$$

Cubic Dirac Equation

$$\left.\begin{aligned} -i\gamma^\mu \partial_\mu \psi + M\psi = (\overline{\psi}\psi)\psi \\ \psi(0) = f \end{aligned}\right\} \qquad \text{on } (t,x) \in \mathbb{R}\times\mathbb{R}^n.$$

here $\overline{\psi} = \psi^\dagger \gamma^0$ is the Dirac adjoint. Also known as Soler Model.

Basic Questions:

1. (LWP) Given data $f \in H^s$ can we find a time $T > 0$ and a unique solution $\psi \in C([0,T], H^s)$ which depends continuously on the data?
2. (GWP and asymptotic behaviour) Can we extend local solution to a global solution $\psi \in C(\mathbb{R}, H^s)$? What happens as $t \to \infty$?
3. (special solutions) Existence/characterisation of soliton solutions? Stability of soliton solutions?

Conserved Quantities

$$\left.\begin{array}{r}-i\gamma^\mu\partial_\mu\psi + M\psi = (\overline{\psi}\psi)\psi \\ \psi(0) = f\end{array}\right\} \qquad \text{on } (t,x)\in\mathbb{R}\times\mathbb{R}^n.$$

- The Charge

$$Q[\psi] = \|\psi\|_{L^2_x}$$

  and the Energy

$$E[\psi] = \int_{\mathbb{R}^n} \frac{i}{2}\big(\overline{\psi}\gamma^0\partial_t\psi - \overline{\partial_t\psi}\gamma^0\psi\big) + \frac{1}{2}\big(\overline{\psi}\psi\big)^2 dx$$

  are conserved under the flow (thus $Q[\psi] = Q[f]$ and $E[\psi] = E[f]$).

Scaling

$$\left.\begin{aligned} -i\gamma^\mu \partial_\mu \psi + M\psi &= (\overline{\psi}\psi)\psi \\ \psi(0) &= f \end{aligned}\right\} \qquad \text{on } (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

- If $M = 0$ and $\psi(t,x)$ is a solution, then $\psi_\lambda(t,x) = \lambda^{\frac{1}{2}}\psi(\lambda t, \lambda x)$ also a solution.
- Rescaling leaves $\dot{H}^{\frac{n-1}{2}}$ norm invariant

$\Rightarrow$ Cubic Dirac equation is critical in $H^{\frac{n-1}{2}}$.

- In particular, we have the following expectation from data $f \in H^s$:

$s \geqslant \frac{n-1}{2}$ problem locally well-posed

$s < \frac{n-1}{2}$ problem ill-posed

Current state of the art $n = 2, 3$: small data gwp and scattering

Theorem (Bejenaru-Herr'15, '16, Bournaveas-C.'15)

Let $n = 2, 3$ and $M \geqslant 0$. There exists $\epsilon > 0$ such that if $\|f\|_{H^{\frac{n-1}{2}}} < \epsilon$ then there exists a global solution $\psi \in C(\mathbb{R}, H^{\frac{n-1}{2}})$ which is unique in a certain subspace, and depends continuously on the data. Moreover ψ scatters to a linear solution as $t \to \pm\infty$, thus there exists $\psi_{\pm\infty}$ with $(-i\gamma^\mu \partial_\mu + M)\psi_{\pm\infty} = 0$ such that

$$\lim_{t \to \pm\infty} \|\psi(t) - \psi_{\pm\infty}(t)\|_{H^{\frac{n-1}{2}}} = 0.$$

- Result also holds in the case of the Thirring Model

$$-i\gamma^\mu \partial_\mu \psi + M\psi = (\overline{\psi}\gamma^\mu \psi)\gamma_\mu \psi.$$

When $n = 3$ can also add combinations of

$$(\overline{\psi}\gamma^5\psi)\psi, \qquad (\overline{\psi}\psi)\gamma^5\psi, \qquad (\overline{\psi}\gamma^5\psi)\gamma^5\psi$$

where $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ (essentially any Lorentz covariant nonlinearity).

Current state of the art $n = 1$: large data gwp and small data modified scattering

Theorem (C.'12, C.-Lindblad'16)

Let $n = 1$ and $M \geqslant 0$. Then cubic Dirac equation is GWP from large data in $L^2(\mathbb{R})$. Moreover, if $M > 0$ and

$$\| \langle x \rangle^4 f \|_{H^5} \ll 1$$

then we have the pointwise asymptotics as $\rho = \sqrt{t^2 - x^2} \to \infty$

$$\psi_1 + \psi_2 = \frac{1}{\sqrt{t-x}} \Big(e^{i\rho} A_+(\rho, \tfrac{x}{t}) + e^{-i\rho} A_-(\rho, \tfrac{x}{t}) \Big) + \mathcal{O}\Big(\frac{1}{\sqrt{(t-x)\rho}} \Big)$$

$$\psi_1 - \psi_2 = \frac{1}{\sqrt{t+x}} \Big(e^{i\rho} A_+(\rho, \tfrac{x}{t}) - e^{-i\rho} A_-(\rho, \tfrac{x}{t}) \Big) + \mathcal{O}\Big(\frac{1}{\sqrt{(t+x)\rho}} \Big)$$

with $A_\pm(\rho, y) = e^{2i|f_\pm(y)| \ln \rho} f_\pm(y)$ and $\psi^T = (\psi_1, \psi_2)$.

- For linear Dirac $A_\pm = f_\pm(y)$ (i.e. no log correction)

Previous Results for Cubic Dirac:

- Local well-posedness for $s > 1$ (subcritical range) due to [ESCOBEDO-VEGA '97].
- Global well-posedness and scattering when $s > 1$ and $M > 0$, or $s = 1$ and some additional angular regularity due to [MACHIHARA-NAKANISHI-OZAWA '03, MACHIHARA-NAKAMURA-OZAWA'04, MACHIHARA-NAKAMURA-NAKANISHI-OZAWA'05].
- If $n = 2$, local well-posedness in subcritical regime [PECHER '14].
- If $n = 1$, gwp for regular data [DELGADO'78], subcritical large data global existence $s > 0$ [SELBERG-TESFAHUN'10],
- Existence of solitary wave solutions [STRAUSS-VÁSZQUEZ '86,CAZENAVE-VÁSZQUEZ '86,MERLE '88, ESTEBAN-SÉRÉ'93, ...]

$$\psi(t, x) = e^{-i\omega t}\psi_\omega(x).$$

- Stability of solitary waves [BOUSAID-CUCCAGNA'12, CONTRERAS-PELINOVSKY-SHIMABUKURO'16, BOUSSAID-COMECH'16...]

Sketch of proof: $n = 2, 3$

Can reduce proof of theorem to constructing Banach space X (for the solution) and N (for nonlinear term) such that

(1) X controls the data space $L^\infty_t H^s$,

$$\|\psi\|_{L^\infty_t H^s(\mathbb{R}^{1+n})} \lesssim \|\psi\|_X.$$

(2) 'Generalised Energy inequality'

$$\|\psi\|_X \lesssim \|\psi(0)\|_{H^s(\mathbb{R}^n)} + \|(-i\gamma^\mu \partial_\mu + M)\psi\|_N.$$

(3) Nonlinear Estimate

$$\|(\overline{\psi}\psi)\psi\|_N \lesssim \|\psi\|^3_X$$

$(1) + (2) + (3) \implies$ well-posedness in H^s.

First Attempt: $n = 3$

Take $X = L^\infty_t H^s_x \cap L^q_t L^\infty_x([0,T] \times \mathbb{R}^3)$ (with $q > 2$) and $N = L^1_t H^s_x([0,T] \times \mathbb{R}^3)$.

- Energy estimate is true (Strichartz type estimates), but nonlinear estimate loses a power of T since

$$\|\overline{\psi}\psi\psi\|_{L^1_T H^s_x} \approx \|\psi^2 \nabla^s \psi\|_{L^1_T L^2_x} \lesssim \|\psi\|^2_{L^2_T L^\infty_x} \|\psi\|_{L^\infty_T H^s_x}$$

  and we only have

$$\|\psi\|_{L^2_t L^\infty_x([0,T]\times\mathbb{R}^3)} \lesssim T^{\frac{1}{2}-\frac{1}{q}} \|\psi\|_X$$

- However does give lwp for $s > 1$ [Escobedo-Vega '97], and can be pushed to give gwp when $s > 1, m > 1$ or have additional angular regularity [Machihara-Nakamura-Nakanishi-Ozawa'05]
- Endpoint case requires $L^2_t L^\infty_x(\mathbb{R}^{1+n})$ bound. Unfortunately this estimate fails [Klainerman-Machedon '93], also fails in $L^2_t(\mathbb{R}, BMO_x(\mathbb{R}^3))$ [Montgomery-Smith '98].
- To improve need two further ingredients:
  1. Null Structure and bilinear estimates (without structure, blow-up can occur [Lindblad'96, D'Ancona-Okamoto'16]).
  2. Replacement for missing $L^2_t L^\infty_x$ estimate.

Null Structure: $n = 1$

Let $-i\gamma^\mu \partial_\mu \psi = 0$ and consider the bilinear term $\overline{\psi}\psi$.

- Solution is of form $\psi(t,x) = \begin{pmatrix} f(x-t) + g(x+t) \\ f(x-t) - g(x+t) \end{pmatrix}$ and

$$\overline{\psi}\psi = 2\Im[f^\dagger(x-t)g(x+t)].$$

- Thus $\overline{\psi}\psi$ is product of waves traveling in different directions!
- Easy consequence:

$$\|\overline{\psi}\psi\|_{L^2_{t,x}} \lesssim \|\psi(0)\|^2_{L^2_x}$$

- Note that

$$\big\||\psi|^2\big\|_{L^2_{t,x}} = \big\||f(x-t)|^2 + |g(x+t)|^2\big\|_{L^2_{t,x}} = \infty.$$

So bilinear $L^2_{t,x}$ estimate fails for generic product.

Null Structure: $n > 1$

Let $-i\gamma^\mu \partial_\mu \psi = 0$ and consider the bilinear term $\overline{\psi}\psi$.

- Introduce potential
$$-i\gamma^\mu \partial_\mu \varphi = \psi.$$
Then $\Box \varphi = 0$ and
$$\overline{\psi}\psi = Q(\varphi, \varphi)$$
where Q is sum of classical null forms
$$Q_{\mu\nu}(u, v) = \partial_\mu u \partial_\nu v - \partial_\nu u \partial_\mu v, \qquad Q_0(u, v) = \partial^\mu u \partial_\nu v$$
- These bilinear forms have improved regularity/decay properties and have been well-studied [Klainerman-Machedon'93], [Klainerman-Foschi'00], [Lee-Vargas'08]
- One consequence:
$$\big\|\overline{\psi}\psi\big\|_{L^2_{t,x}} \lesssim \big\|\psi(0)\big\|_{L^2_x} \|\psi(0)\|_{H^{\frac{n-1}{2}}}$$
- Again estimate fails for generic products like $|\psi|^2$.

Missing $L^2_t L^\infty_x$ Strichartz estimate: The Knapp example

Let $f \in L^2_x$ with

$$\operatorname{supp} \widehat{f} \subset \{|\xi| \approx \lambda,\ \theta(\xi, \omega) \leqslant \alpha\}, \qquad \omega \in \mathbb{S}^{n-1}$$

- Corresponding solution to wave equation is

$$\big(e^{-it\sqrt{-\Delta}} f\big)(x) = \int_{\mathbb{R}^n} e^{-it|\xi|} \widehat{f}(\xi) e^{ix\cdot\xi} d\xi = \int_{\mathbb{R}^n} e^{-it\left(|\xi| - \xi\cdot\omega\right)} \widehat{f}(\xi) e^{i(x - t\omega)\cdot\xi} d\xi.$$

- If $|t| \ll \lambda\alpha^{-2}$ then

$$t\big(|\xi| - \xi\cdot\omega\big) \ll 1 \qquad \Longrightarrow \qquad e^{-it(|\xi| - \omega\cdot\xi)} \approx 1$$

  and therefore solution is essentially traveling wave

$$e^{-it\sqrt{-\Delta}} f(x) \approx f(x - t\omega)$$

- For Knapp example, dispersion does not kick in until times $|t| \gtrsim \lambda\alpha^{-2}$. In particular, for α small, times can be very large...

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Knap Example and Null Frames

Let $f \in L^2_x$ with

$$\operatorname{supp} \widehat{f} \subset \{|\xi| \approx \lambda,\ \theta(\xi, \omega) \leqslant \alpha\}.$$

- For times $|t| \ll T = \lambda\alpha^{-2}$, we have seen that

$$e^{-it\sqrt{-\Delta}} f(x) \approx f(x - t\omega).$$

- In particular, this implies that

$$\|\chi(\tfrac{t}{T}) e^{-it\sqrt{-\Delta}} f\|_{L^2_t L^\infty_x} \approx T^{\frac{1}{2}} \|f\|_{L^\infty_x} \lesssim (\alpha\lambda)^{\frac{n-1}{2}} \frac{\lambda}{\alpha} \|f\|_{L^2_x}.$$

- On the other hand, if we introduce the null frame

$$t_\omega = t + x \cdot \omega, \qquad x_\omega = x - \frac{1}{2}(t + x \cdot \omega)\omega$$

  then computation gives

$$\|\chi(\tfrac{t}{T}) e^{-it\sqrt{-\Delta}} f\|_{L^2_{t_\omega} L^\infty_{x_\omega}} \lesssim (\alpha\lambda)^{\frac{n-1}{2}} \|f\|_{L^2_x}$$

  which is much smaller!!

Knap Example and Null Frames

Let $f \in L^2_x$ with

$$\operatorname{supp} \widehat{f} \subset \{|\xi| \approx \lambda,\ \theta(\xi,\omega) \leqslant \alpha\}.$$

$$t_\omega = t + x\cdot\omega, \qquad x_\omega = x - \frac{1}{2}(t + x\cdot\omega)\omega$$

- Moral: $L^2_t L^\infty_x$ bad, but if we adapt frame to f, the we do have acceptable $L^2_{t_\omega} L^\infty_{x_\omega}$ bounds.
- This idea was introduced by Tataru in work on wave maps [Tataru'01], and played a crucial role in resolving the small data global well-posedness theory.
- Compare to $n = 1$ case: solution to wave equation given by traveling waves

  $$f(x-t) + g(x+t)$$

  Clearly if $\|f(x-t)\|_{L^2_t L^\infty_x} = \infty$ but $\|f(x-t)\|_{L^2_{x-t} L^\infty_{x+t}} = \|f\|_{L^2_x}$.
- Exploits focusing property of the wave, rather than dispersion.

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Replacement for $L^2_t L^\infty_x$: The plane wave space $PW(\kappa)$

- Let $\kappa \subset \mathbb{S}^{n-1}$ be a cap on the sphere (so a set of directions). Then $\phi(t,x)$ is a $PW(\kappa)$ atom if there exists $\omega \in 2\kappa$ such that

$$\|\phi\|_{L^2_{t_\omega} L^\infty_{x_\omega}} \leqslant 1.$$

- We then define $PW(\kappa)$ to be the corresponding atomic Banach space

$$PW(\kappa) = \Big\{ \sum_j c_j \phi_j \,\Big|\, (c_j) \in \ell^1(\mathbb{N}), \quad \phi_j \text{ is a } PW(\kappa) \text{ atom} \Big\}.$$

- Basic Properties:
  - $PW(\kappa)$ is a Banach space
  - Given any $\omega \in \kappa$, have bound

$$\|u\|_{PW(\kappa)} \leqslant \|u\|_{L^2_{t_\omega} L^\infty_{x_\omega}}$$

$\implies$ have freedom to adapt frame to the function u

Summary: $n = 2, 3$

- $PW(\kappa)$ spaces form a suitable replacement for missing $L^2_t L^\infty_x$ estimate.

- Price you have to pay is you need energy estimates in null frames, usually gives a loss, but for cubic Dirac null structure gives additional cancellation.

- Need bilinear estimates in $L^2_{t,x}$, exploits null structure to get sharp bounds.

- Small data global well-posedness follows by iterating in spaces base on $PW(\kappa)$, Strichartz type norms, $X^{s,b}$ norms, and $L^\infty_{t_\omega} L^2_{x_\omega}$ type norms.

Proof: $n = 1$

- Again, key ingredients are bilinear null form estimates, and null frames (which now take the simple form of null coordinates $t \pm x$).

- Conservation of charge, together with a nonconcentration argument (charge cannot concentrate at a point), implies large data result.

- Proof of scattering requires weighted energy estimates across hyperboloids $\tau = t^2 - x^2$. Idea is to remove expected linear behaviour, control error estimates using energy estimates, and reduce to an ODE of the form

$$\partial_\tau U = \frac{i}{\tau}|U|^2 U$$

  which gives the log correction (strategy has been used in earlier work on Klein-Gordon equations [DELORT'01], Schrodinger equation [LINDBLAD-SOFFER'05]).

Dirac-Klein-Gordon system: previous results in $n = 3$.

$$-i\gamma^\mu \partial_\mu \psi + M\psi = \phi\psi$$
$$\Box\phi + m^2\phi = \overline{\psi}\psi$$

the scalar $\phi : \mathbb{R}^{1+n} \to \mathbb{R}$ and spinor $\psi : \mathbb{R}^{1+n} \to \mathbb{C}^N$. Masses satisfy $M, m \geqslant 0$.

- Scaling is $\big(\psi(0), \phi(0), \partial_t\phi(0)\big) \in L^2 \times H^{\frac{1}{2}} \times H^{-\frac{1}{2}}$.
- Local well-posedness is known in $H^\epsilon \times H^{\frac{1}{2}+\epsilon} \times H^{-\frac{1}{2}+\epsilon}$ [D'Ancona-Foschi-Selberg'07]. Builds on earlier work of [Klainerman-Machedon'94] [Beals-Bezard'96] [Bournaveas'99] [Fang-Grillakis'05].
- Global well-posedness for large "symmetric" data holds [Chadam-Glassey'74], [Bachelot'89], [Dias-Figueira'91].

Dirac-Klein-Gordon system: previous results in $n = 3$.

$$-i\gamma^\mu \partial_\mu \psi + M\psi = \phi\psi$$
$$\Box\phi + m^2\phi = \overline{\psi}\psi$$

- If $2M > m$ or $M = m = 0$, have global well-posedness and scattering for small data in $\langle \Omega \rangle^{-1} \dot{B}^0_{2,1} \times \langle \Omega \rangle^{-1} \dot{B}^{\frac{1}{2}}_{2,1} \times \langle \Omega \rangle^{-1} \dot{B}^{-\frac{1}{2}}_{2,1}$ [WANG'13] with
$$\langle \Omega \rangle = 1 + (-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}.$$
- If $2M > m$, have global well-posedness and scattering for small data in $H^\epsilon \times H^{\frac{1}{2}+\epsilon} \times H^{-\frac{1}{2}+\epsilon}$ [BEJENARU-HERR'14].

Scattering in resonant region

$$-i\gamma^\mu \partial_\mu \psi + M\psi = \phi\psi$$
$$\Box\phi + m^2\phi = \overline{\psi}\psi$$

Theorem (C.-Herr'16)

Let $2M > m > 0$ and $\sigma > 0$, or $0 < 2M \leqslant m$ and $\sigma > \frac{30}{7}$. Then DKG is globally well-posed and scatters for small data in $\langle\Omega\rangle^{-\sigma} L^2_x \times \langle\Omega\rangle^{-\sigma} H^{\frac{1}{2}}_x \times \langle\Omega\rangle^{-\sigma} H^{-\frac{1}{2}}_x$.

- Endpoint result would be $\sigma = 0$ (i.e no angular derivatives)
- Main additional difficulty is presence of resonant interactions in nonlinearity (i.e. $(\overline{\psi}\psi)$ behaves like linear solution).

Resonant Regions

$$-i\gamma^\mu \partial_\mu \psi + M\psi = \phi\psi$$
$$\Box\phi + m^2\phi = \overline{\psi}\psi$$

A computation gives

$$2M > m \implies \text{no resonances}$$
$$2M = m \implies \text{resonances but cancelled by null structure}$$
$$2M < m \implies \text{resonances } \textit{not} \text{ cancelled by null structure.}$$

However: If $2M < m$ we have transversality on resonant set

$\implies$ can exploit this via bilinear restriction estimates.

Transference in V^2

Theorem (C.-Herr'16)

Let $p > \frac{n+3}{n+1}$. Let $\Lambda_j \subset \{-\frac{1}{4} < |\xi| < 4\}$ and $\Phi_j : \Lambda_j \to \mathbb{R}$ be smooth phases such that for every $\xi \in \Lambda_1$, $\eta \in \Lambda_2$ we have the transversality condition

$$|\nabla \Phi_1(\xi) - \nabla \Phi_2(\eta)| \gtrsim 1$$

and a "curvature" condition holds. If $\operatorname{supp} \widehat{u} \subset \Lambda_1$ and $\operatorname{supp} \widehat{v} \subset \Lambda_2$ we have

$$\|uv\|_{L^p_{t,x}(\mathbb{R}^{1+n})} \lesssim \|u\|_{V^2_{\Phi_1}} \|v\|_{V^2_{\Phi_2}}.$$

Here $\|u\|_{V^2_\Phi} = \|u\|_{L^\infty_t L^2_x} + |u|_{V^2_\Phi}$ with

$$|u|_{V^2_\Phi} = \sup_{t_j \in \mathcal{Z}} \Big(\sum_j \|e^{-it\Phi(-i\nabla)} u(t_j) - e^{-it\Phi(-i\nabla)} u(t_{j-1})\|^2_{L^2_x} \Big)^{\frac{1}{2}}$$

and $\mathcal{Z} = \{... < t_j < t_{j+1} < t_{j+2} < ...\}$ is collection of all increasing sequences. The spaces V^p and U^p first introduced by Tataru in unpublished work on wave maps, studied systematically in [Koch-Tataru'05] [Koch-Tataru'07].

Transference in V^2

Theorem (C.-Herr'16)

Let $p > \frac{n+3}{n+1}$. Let $\Lambda_j \subset \{-\frac{1}{4} < |\xi| < 4\}$ and $\Phi_j : \Lambda_j \to \mathbb{R}$ be smooth phases such that for every $\xi \in \Lambda_1$, $\eta \in \Lambda_2$ we have the transversality condition

$$|\nabla\Phi_1(\xi) - \nabla\Phi_2(\eta)| \gtrsim 1$$

and a "curvature" condition holds. If $\operatorname{supp}\widehat{u} \subset \Lambda_1$ and $\operatorname{supp}\widehat{v} \subset \Lambda_2$ we have

$$\|uv\|_{L^p_{t,x}(\mathbb{R}^{1+n})} \lesssim \|u\|_{V^2_{\Phi_1}} \|v\|_{V^2_{\Phi_2}}.$$

- Homogeneous case $u = e^{it\Phi_1(-i\nabla)}f$, $v = e^{it\Phi_2(-i\nabla)}$ due to [Tao-Vargas-Vega'98, Wolff'01, Tao'03, Lee-Vargas'10, Bejenaru'16...]
- Key point is that range of p improves significantly in presence of transversality.
- Interpolating with Strichartz estimates gives result for $p < 2$ close to 2 [Sterbenz-Tataru'10]. Full range of p requires adapting induction on scales arguments of Tao and Wolff to V^2_Φ.

Open Problems

1. Endpoint DKG: Cubic Dirac the small data theory is well-understood. Some gaps remain for the DKG equation:

   Do we have GWP for DKG in $L^2 \times H^{\frac{1}{2}} \times H^{-\frac{1}{2}}$?

2. "Medium" Data Cubic Dirac: If $n = 3$, then the Dirac equation is energy supercritical, thus no obvious reason why generic large data solutions should stay bounded.

   Can we construct blow-up solutions for cubic Dirac?

   A related question relates to a ground state, can we find a minimum (in some version of "energy" or norm) soliton?

3. Scattering for Cubic Dirac in $n = 1$: The modified scattering result is far from optimal. Natural question is if we can extend result to L^2. First step:

   Is there a way to characterise/understand the scattering state in L^2?