On the spectral properties of Dirac operators with electrostatic δ-shell interactions

Markus Holzmann

Graz University of Technology
joint work with J. Behrndt, P. Exner, and V. Lotoreichik

Linear and Nonlinear Dirac Equation: advances and open problems, Como, February 10, 2017

Motivation

Object of interest:

$$A_\eta := A_0 + \eta\delta_\Sigma$$

in $L^2(\mathbb{R}^3;\mathbb{C}^4)$, where

- $\eta \in \mathbb{R}$
- A_0 is the free Dirac operator
- $\Sigma \subset \mathbb{R}^3$ is the boundary of a bounded C^2-smooth domain

Questions:

- Rigorous mathematical definition of A_η as self-adjoint operator
- Spectral properties of A_η
- Scattering theory for A_η
- Justification for the usage of A_η

Motivation for extension theory

$$A_\eta := A_0 + \eta\delta_\Sigma,$$

Observations:

- $A_\eta f(x) = A_0 f(x)$, if $x \notin \Sigma$
- δ-shell interaction is modelled by jump condition at Σ
- Introduce A_η as self-adjoint extension of

$$S := A_0 \upharpoonright H_0^1(\mathbb{R}^3 \setminus \Sigma; \mathbb{C}^4)$$

Extension theoretical approaches:

- Arrizabalaga, Mas, Vega, ...
  - designed for application to the Dirac operator
- Quasi boundary triples:
  - more abstract approach
  - successfully applied for Schrödinger operators with δ-interactions

Outline

1. Motivation

2. Quasi boundary triples, their Weyl functions and Dirac

3. Dirac operators with electrostatic δ-shell interactions
   - Dirac operators with δ-interactions of strength $\eta \neq \pm 2c$
   - Dirac operators with δ-interactions of strength $\eta = \pm 2c$

4. Summary and outlook

Boundary triples – a bit of history

Ordinary boundary triples

- abstract approach in extension theory for symmetric operators
- introduced in the 1970s (Bruk, Kochubei)
- far developed, used in many applications
- some names: Behrndt, Cacciapuoti, de Snoo, Derkach, Geyler, Malamud, Neidhardt, Pankrashkin, Posilicano, ...

Quasi boundary triples

- generalization of ordinary boundary triples
- introduced by Behrndt and M. Langer in 2007 to study partial differential operators with special boundary/interface conditions
- there exist several similar concepts
- some names: Behrndt, Hassi, M. Langer, Lotoreichik, Malamud, Posilicano, Rohleder, ...
- related to approach of Arrizabalaga, Mas, and Vega

Quasi boundary triples – definition

Definition

Assumptions:

- $\mathcal{H}$ and $\mathcal{G}$ are Hilbert spaces
- S is a closed symmetric operator in $\mathcal{H}$
- T is an operator such that $\overline{T} = S^*$

$\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called a quasi boundary triple, if $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G}$

- $\operatorname{ran}(\Gamma_0, \Gamma_1)$ is dense in $\mathcal{G} \times \mathcal{G}$;
- $A_0 := T \restriction \ker \Gamma_0$ is self-adjoint;
- $\forall f, g \in \operatorname{dom} T$:

$$(Tf, g)_{\mathcal{H}} - (f, Tg)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}}$$

Goal: construct quasi boundary triple for the Dirac operator

The free Dirac operator

- Define

$$A_0 f := -ic \sum_{j=1}^{3} \alpha_j \partial_j f + mc^2 \beta f, \quad \text{dom}\, A_0 = H^1(\mathbb{R}^3; \mathbb{C}^4)$$

- m is the mass of the particle and c the speed of light
- $\alpha_1, \alpha_2, \alpha_3$ and β are the Dirac matrices

$$\alpha_j := \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad \text{and} \quad \beta := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

where σ_j are the Pauli spin matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A quasi boundary triple for the Dirac operator

- Let Ω_+ be a C^2-smooth bdd. domain, $\Sigma := \partial\Omega_+$, $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega_+}$
- Notation: for $f \in L^2(\mathbb{R}^3)$ we write $f_\pm = f|_{\Omega_\pm}$
- Define $S := A_0 \upharpoonright H_0^1(\mathbb{R}^3 \setminus \Sigma)$
- Then:

$$S^* f = (-ic\alpha \cdot \nabla + mc^2\beta) f_+ \oplus (-ic\alpha \cdot \nabla + mc^2\beta) f_-$$
$$\mathrm{dom}\, S^* = \left\{ f = f_+ \oplus f_- : (-ic\alpha \cdot \nabla + mc^2\beta) f_\pm \in L^2(\Omega_\pm) \right\}$$

- Introduce

$$Tf = (-ic\alpha \cdot \nabla + mc^2\beta) f_+ \oplus (-ic\alpha \cdot \nabla + mc^2\beta) f_-$$
$$\mathrm{dom}\, T = H^1(\mathbb{R}^3 \setminus \Sigma) := H^1(\Omega_+) \oplus H^1(\Omega_-)$$

- It holds $\overline{T} = S^*$

A quasi boundary triple for the Dirac operator

- Define for $f = f_+ \oplus f_- \in H^1(\Omega_+) \oplus H^1(\Omega_-)$

$$\Gamma_0 f = ic\alpha \cdot \nu(f_+|_\Sigma - f_-|_\Sigma) \quad \text{and} \quad \Gamma_1 f = \frac{1}{2}(f_+|_\Sigma + f_-|_\Sigma)$$

  (ν is the outer unit normal vector field for Ω_+)
- Note: $\Gamma_0 f, \Gamma_1 f \notin L^2(\Sigma)$ for $f \in \operatorname{dom} S^*$

Theorem

S is closed and symmetric and $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for $\overline{T} = S^*$ such that $T \upharpoonright \ker \Gamma_0$ is the free Dirac operator A_0.

Proof:

- Recall: for $f = f_+ \oplus f_- \in H^1(\Omega_+) \oplus H^1(\Omega_-)$

$$\Gamma_0 f = ic\alpha \cdot \nu (f_+|_\Sigma - f_-|_\Sigma) \quad \text{and} \quad \Gamma_1 f = \frac{1}{2}(f_+|_\Sigma + f_-|_\Sigma)$$

- $\operatorname{ran}(\Gamma_0, \Gamma_1) = H^{1/2}(\Sigma) \times H^{1/2}(\Sigma)$ is dense in $L^2(\Sigma) \times L^2(\Sigma)$
- $\ker \Gamma_0 = H^1(\mathbb{R}^3) \Rightarrow T \upharpoonright \ker \Gamma_0 = A_0$
- Integration by parts in $\Omega_\pm$

$$\big((-ic\alpha \cdot \nabla + mc^2\beta) f_\pm, g_\pm\big)_{\Omega_\pm} - \big(f_\pm, (-ic\alpha \cdot \nabla + mc^2\beta) g_\pm\big)_{\Omega_\pm} = \pm\big(- ic\alpha \cdot \nu f_\pm|_\Sigma, g_\pm|_\Sigma\big)_\Sigma$$

- $$(Tf, g)_{\mathbb{R}^3} - \big(f, Tg\big)_{\mathbb{R}^3} = (\Gamma_1 f, \Gamma_0 g)_\Sigma - (\Gamma_0 f, \Gamma_1 g)_\Sigma$$

γ-field and Weyl function

- For $\lambda \in \rho(A_0)$ it holds that

$$\operatorname{dom} T = \operatorname{dom} A_0 \dot{+} \ker(T - \lambda) = \ker \Gamma_0 \dot{+} \ker(T - \lambda)$$

- The mapping $\Gamma_0 \upharpoonright \ker(T - \lambda)$ is injective for $\lambda \in \rho(A_0)$

Definition

Define for $\lambda \in \rho(A_0)$ the mappings

(i) $\gamma(\lambda) := (\Gamma_0 \upharpoonright \ker(T - \lambda))^{-1}$ … γ-field

(ii) $M(\lambda) := \Gamma_1 \gamma(\lambda) = \Gamma_1(\Gamma_0 \upharpoonright \ker(T - \lambda))^{-1}$ … Weyl function

- $\gamma(\lambda)$ maps $\varphi \in \operatorname{ran} \Gamma_0$ to a solution u_λ of $(T - \lambda)u_\lambda = 0$, $\Gamma_0 u_\lambda = \varphi \Rightarrow \gamma(\lambda)$ is a kind of Poisson operator
- $M(\lambda)$ … abstract Dirichlet-to-Neumann map
- $\gamma(\lambda)^* = \Gamma_1(A_0 - \overline{\lambda})^{-1}$

γ-field and Weyl function for our triple

Recall:

- $A_0 = T \upharpoonright \ker \Gamma_0$ is the free Dirac operator
- $\sigma(A_0) = (-\infty, -mc^2] \cup [mc^2, \infty)$
- For $\lambda \in \rho(A_0)$ it holds

$$(A_0 - \lambda)^{-1} f(x) = \int_{\mathbb{R}^3} G_\lambda(x-y) f(y) \mathrm{d}y, \quad x \in \mathbb{R}^3,$$

with a known function G_λ

$$G_\lambda(x) = \left(\frac{\lambda}{c^2} I + m\beta + \left(1 - i\sqrt{\frac{\lambda^2}{c^2} - (mc)^2} |x| \right) \frac{i(\alpha \cdot x)}{c|x|^2} \right) \frac{e^{i\sqrt{\lambda^2/c^2 - (mc)^2}|x|}}{4\pi |x|}.$$

γ-field and Weyl function for our triple

Recall:

- $A_0 = T \upharpoonright \ker \Gamma_0$ is the free Dirac operator
- $\sigma(A_0) = (-\infty, -mc^2] \cup [mc^2, \infty)$
- For $\lambda \in \rho(A_0)$ it holds

$$(A_0 - \lambda)^{-1} f(x) = \int_{\mathbb{R}^3} G_\lambda(x-y) f(y) \mathrm{d}y, \quad x \in \mathbb{R}^3,$$

  with a known function G_λ
- It holds $\gamma(\lambda)^* = \Gamma_1 (A_0 - \overline{\lambda})^{-1} : L^2(\mathbb{R}^3) \to L^2(\Sigma)$,

$$\gamma(\lambda)^* f(x) := \int_{\mathbb{R}^3} G_{\overline{\lambda}}(x-y) f(y) \mathrm{d}y, \qquad x \in \Sigma.$$

γ-field and Weyl function for our triple

- $\gamma(\lambda)^* = \Gamma_1(A_0 - \overline{\lambda})^{-1} : L^2(\mathbb{R}^3) \to L^2(\Sigma)$,

$$\gamma(\lambda)^* f(x) := \int_{\mathbb{R}^3} G_{\overline{\lambda}}(x-y) f(y) \mathrm{d}y, \qquad x \in \Sigma.$$

- $\operatorname{ran} \Gamma_0 = H^{1/2}(\Sigma)$ (because $\Gamma_0 f = ic\alpha \cdot \nu(f_+|_\Sigma - f_-|_\Sigma)$)

Proposition

Let $\lambda \in \rho(A_0)$. Then:

(i) $\gamma(\lambda) : H^{1/2}(\Sigma) \to L^2(\mathbb{R}^3)$,

$$\gamma(\lambda)\varphi(x) := \int_\Sigma G_\lambda(x-y)\varphi(y) \mathrm{d}\sigma(y), \qquad x \in \mathbb{R}^3.$$

(ii) $M(\lambda) : H^{1/2}(\Sigma) \to L^2(\Sigma)$,

$$M(\lambda)\varphi(x) = \lim_{\varepsilon \searrow 0} \int_{|x-y|>\varepsilon} G_\lambda(x-y)\varphi(y) \mathrm{d}\sigma(y), \quad x \in \Sigma.$$

γ-field and Weyl function for our triple

Proposition

Let $\lambda \in \rho(A_0)$. Then there exist continuous extensions

(i) $\overline{\gamma(\lambda)} : L^2(\Sigma) \to L^2(\mathbb{R}^3)$,

$$\overline{\gamma(\lambda)}\varphi(x) := \int_\Sigma G_\lambda(x-y)\varphi(y)\mathrm{d}\sigma(y), \qquad x \in \mathbb{R}^3.$$

(ii) $\gamma(\lambda)^* : L^2(\mathbb{R}^3) \to L^2(\Sigma)$,

$$\gamma(\lambda)^* f(x) := \int_{\mathbb{R}^3} G_{\overline{\lambda}}(x-y)f(y)\mathrm{d}y, \qquad x \in \Sigma.$$

(iii) $\overline{M(\lambda)} : L^2(\Sigma) \to L^2(\Sigma)$,

$$\overline{M(\lambda)}\varphi(x) = \lim_{\varepsilon \searrow 0} \int_{|x-y|>\varepsilon} G_\lambda(x-y)\varphi(y)\mathrm{d}\sigma(y), \quad x \in \Sigma.$$

Krein-type resolvent formula

- Let $\Theta : \mathcal{G} \to \mathcal{G}$ be a symmetric operator
- Define $A_\Theta := T \upharpoonright \ker(\Theta\Gamma_0 - \Gamma_1)$
- Green's identity: A_Θ is symmetric

Theorem (Behrndt, M. Langer 2007)

Let $\lambda \in \rho(A_0)$.

(i) $\lambda \in \sigma_p(A_\Theta) \Leftrightarrow 0 \in \sigma_p(\Theta - M(\lambda))$;

(ii) *If* $\lambda \notin \sigma_p(A_\Theta)$, *then* $f \in \operatorname{ran}(A_\Theta - \lambda) \Leftrightarrow \gamma(\overline{\lambda})^* f \in \operatorname{ran}(\Theta - M(\lambda))$;

(iii) *If* $\lambda \notin \sigma_p(A_\Theta)$, *then for* $f \in \operatorname{ran}(A_\Theta - \lambda)$

$$(A_\Theta - \lambda)^{-1} f = (A_0 - \lambda)^{-1} f + \gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\overline{\lambda})^* f;$$

Krein-type resolvent formula

- Let $\Theta : \mathcal{G} \to \mathcal{G}$ be a symmetric operator
- Define $A_\Theta := T \upharpoonright \ker(\Theta\Gamma_0 - \Gamma_1)$
- Green's identity: A_Θ is symmetric

Theorem (Behrndt, M. Langer 2007)

Let $\lambda \in \rho(A_0)$.

(i) $\lambda \in \sigma_p(A_\Theta) \Leftrightarrow 0 \in \sigma_p(\Theta - M(\lambda))$;

(ii) *If $\lambda \notin \sigma_p(A_\Theta)$, then $f \in \operatorname{ran}(A_\Theta - \lambda) \Leftrightarrow \gamma(\overline{\lambda})^* f \in \operatorname{ran}(\Theta - M(\lambda))$;*

(iii) *If $\lambda \notin \sigma_p(A_\Theta)$, then for $f \in \operatorname{ran}(A_\Theta - \lambda)$*

$$(A_\Theta - \lambda)^{-1} f = (A_0 - \lambda)^{-1} f + \overline{\gamma(\lambda)} \big(\Theta - \overline{M(\lambda)}\big)^{-1} \gamma(\overline{\lambda})^* f;$$

(iv) *If $\lambda \notin \sigma_p(A_\Theta)$ and $\operatorname{ran} \gamma(\overline{\lambda})^* \subset \operatorname{ran}(\Theta - M(\lambda))$, then $\lambda \in \rho(A_\Theta)$.*

Dirac operators with δ-shell interactions

Recall:

$$Tf = (-ic\alpha \cdot \nabla + mc^2\beta)f_+ \oplus (-ic\alpha \cdot \nabla + mc^2\beta)f_-$$
$$\operatorname{dom} T = H^1(\mathbb{R}^3 \setminus \Sigma) := H^1(\Omega_+) \oplus H^1(\Omega_-),$$

$$\Gamma_0 f = ic\alpha\cdot\nu(f_+|_\Sigma - f_-|_\Sigma) \quad \text{and} \quad \Gamma_1 f = \frac{1}{2}(f_+|_\Sigma + f_-|_\Sigma), \quad f \in \operatorname{dom} T$$

Definition

For $\eta \in \mathbb{R} \setminus \{0\}$ define

$$A_\eta := T \restriction \ker\left(-\tfrac{1}{\eta}\Gamma_0 - \Gamma_1\right),$$
$$\operatorname{dom} A_\eta = \left\{f \in \operatorname{dom} T : ic\alpha \cdot \nu(f_+|_\Sigma - f_-|_\Sigma) = -\tfrac{\eta}{2}(f_+|_\Sigma + f_-|_\Sigma)\right\}.$$

Observe: $\Theta = -\frac{1}{\eta}$

Case $\eta \neq \pm 2c$: Basic results

Theorem (Vega et al.; Behrndt, Exner, H., Lotoreichik)

Let $\eta \neq \pm 2c$. Then:

(i) $A_\eta = A_\eta^*$ *and for* $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$(A_\eta - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(I + \eta M(\lambda)\big)^{-1} \eta \gamma(\overline{\lambda})^*;$$

(ii) $\operatorname{dom} A_\eta \subset H^1(\Omega_+) \oplus H^1(\Omega_-)$*;*

(iii) $\sigma_{ess}(A_\eta) = \sigma_{ess}(A_0) = (-\infty, -mc^2] \cup [mc^2, \infty)$*.*

(iv) $\sigma(A_\eta) \cap (-mc^2, mc^2)$ *is finite;*

(v) $\sigma(A_\eta) \cap (-mc^2, mc^2) = \emptyset$ *for* $|\eta|$ *too big or too small.*

Sketch of the proof

Ad (i)

- We show $\operatorname{ran}(A_\eta - \lambda) = L^2(\mathbb{R}^3)$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$
- $f \in \operatorname{ran}(A_\eta - \lambda) \Leftrightarrow \gamma(\overline{\lambda})^* f \in \operatorname{ran}(-\frac{1}{\eta} I - M(\lambda))$
- $$\begin{aligned}\operatorname{ran}(-\tfrac{1}{\eta} I - M(\lambda)) &\supset \operatorname{ran}\left[(-\tfrac{1}{\eta} I - M(\lambda))(\tfrac{1}{\eta} I - M(\lambda))\right] \\ &= \operatorname{ran}\left(\tfrac{1}{\eta^2} I - M(\lambda)^2\right)\end{aligned}$$
- $M(\lambda)^2 = \frac{1}{4c^2} I + K(\lambda)$ with $K(\lambda)$ compact in $H^{1/2}(\Sigma)$
- Fredholm:

$$\operatorname{ran}(-\tfrac{1}{\eta} I - M(\lambda)) \supset \operatorname{ran}\left(\left(\tfrac{1}{\eta^2} - \tfrac{1}{4c^2}\right) I - K(\lambda)\right) = H^{1/2}(\Sigma)$$

- $\operatorname{ran} \gamma(\overline{\lambda})^* = H^{1/2}(\Sigma) \subset \operatorname{ran}(-\frac{1}{\eta} I - M(\lambda))$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$

(iv) $\lambda \in \sigma_p(A_\eta) \Leftrightarrow 0 \in \sigma_p\big(-\frac{1}{\eta} I - M(\lambda)\big)$ + properties of $M(\lambda)$

The nonrelativistic limit

Theorem (Behrndt, Exner, H., Lotoreichik)

Let $\eta \in \mathbb{R}$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then,

$$\lim_{c \to \infty} \left(A_\eta - (\lambda + mc^2)\right)^{-1} = \left(-\tfrac{1}{2m}\Delta + \eta\delta_\Sigma - \lambda\right)^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix},$$

in the operator norm.

Discussion:

- Justification for the usage of A_η;
- Convergence in norm resolvent sense: $\sigma(A_\eta)$ and $\sigma\big(-\frac{1}{2m}\Delta + \eta\delta_\Sigma + mc^2\big)$ are approximately the same for large c.

The nonrelativistic limit

Theorem (Behrndt, Exner, H., Lotoreichik)

Let $\eta \in \mathbb{R}$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then,

$$\lim_{c \to \infty} \left(A_\eta - (\lambda + mc^2)\right)^{-1} = \left(-\frac{1}{2m}\Delta + \eta\delta_\Sigma - \lambda\right)^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix},$$

in the operator norm.

Sketch of the proof:

- Krein's resolvent formula:

$$\begin{aligned}(A_\eta - (\lambda + mc^2))^{-1} &= (A_0 - (\lambda + mc^2))^{-1} \\ &\quad - \overline{\gamma(\lambda + mc^2)}\left(I + \eta\overline{M(\lambda + mc^2)}\right)^{-1}\eta\gamma(\overline{\lambda} + mc^2)^*;\end{aligned}$$

- Compute the limits of $(A_0 - (\lambda + mc^2))^{-1}$, $\overline{\gamma(\lambda + mc^2)}$, $\overline{M(\lambda + mc^2)}$ and $\gamma(\overline{\lambda} + mc^2)^*$, as $c \to \infty$.

Case $\eta = \pm 2c$: Essential self-adjointness

Theorem (Behrndt, H.; Ourmieres-Bonafos, Vega)

$A_{\pm 2c}$ is essentially self-adjoint.

Sketch of the proof (for $\eta = 2c$):

- We show that $\operatorname{ran}(A_{2c} - \lambda)$ is dense in $L^2(\mathbb{R}^3)$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$
- $f \in \operatorname{ran}(A_{2c} - \lambda) \Leftrightarrow \gamma(\overline{\lambda})^* f \in \operatorname{ran}(-\frac{1}{2c} I - M(\lambda))$
- $\operatorname{ran} \gamma(\overline{\lambda})^* = \operatorname{ran} \left(\Gamma_1 (A_0 - \lambda)^{-1}\right) = H^{1/2}(\Sigma)$
- As before

$$\operatorname{ran}(-\tfrac{1}{2c} I - M(\lambda)) \supset \operatorname{ran} \left(\tfrac{1}{4c^2} I - \tfrac{1}{4c^2} I - K(\lambda)\right) = \operatorname{ran} K(\lambda)$$

- $K(\lambda)$ is compact in $H^{1/2}(\Sigma)$, injective and has dense range in $H^{1/2}(\Sigma)$

Case $\eta = \pm 2c$: Self-adjoint realization

- We know: $A_{\pm 2c} = T \upharpoonright \ker\left(\mp \frac{1}{2c}\Gamma_0 - \Gamma_1\right)$ is essentially self-adjoint
- In general: $\overline{A_{\pm 2c}} \not\subset T$
- Define $\overline{A_{\pm 2c}}$ as restriction of $S^* = \overline{T}$,

$$S^* f = (-ic\alpha \cdot \nabla + mc^2\beta)f_+ \oplus (-ic\alpha \cdot \nabla + mc^2\beta)f_-$$
$$\operatorname{dom} S^* = \left\{f = f_+ \oplus f_- : (-ic\alpha \cdot \nabla + mc^2\beta)f_\pm \in L^2(\Omega_\pm)\right\}$$

Case $\eta = \pm 2c$: Self-adjoint realization

- We know: $A_{\pm 2c} = T \upharpoonright \ker\left(\mp \frac{1}{2c}\Gamma_0 - \Gamma_1\right)$ is essentially self-adjoint
- In general: $\overline{A_{\pm 2c}} \not\subset T$
- Define $\overline{A_{\pm 2c}}$ as restriction of $S^* = \overline{T}$,

Proposition (Behrndt, Micheler 2014)

$\Gamma_0, \Gamma_1 : \mathrm{dom}\, T \to L^2(\Sigma)$ *have surjective extensions*
$\widetilde{\Gamma}_0, \widetilde{\Gamma}_1 : \mathrm{dom}\, S^* \to H^{-1/2}(\Sigma)$

Theorem (Behrndt, H.; Ourmieres-Bonafos, Vega)

$\overline{A_{\pm 2c}} = S^* \upharpoonright \ker\left(\widetilde{\Gamma}_0 \pm 2c\widetilde{\Gamma}_1\right)$, *i.e.*

$$\mathrm{dom}\, S^* \ni f \in \mathrm{dom}\, \overline{A_{\pm 2c}} \Leftrightarrow \mp\frac{1}{2c}\widetilde{\Gamma}_0 f = \widetilde{\Gamma}_1 f \quad \textit{in } H^{-1/2}(\Sigma)$$

Properties of $\overline{A_{\pm 2c}}$

Recall: $\overline{A_{\pm 2c}} = S^* \upharpoonright \ker\left(\mp \frac{1}{2c}\widetilde{\Gamma}_0 - \widetilde{\Gamma}_1\right)$

Proposition (Behrndt, Micheler 2014)

Let $\lambda \in \rho(A_0)$. Then, $\gamma(\lambda)$ and $M(\lambda)$ have continuous extensions

$$\widetilde{\gamma}(\lambda) : H^{-1/2}(\Sigma) \to L^2(\mathbb{R}^3),$$

$$\widetilde{M}(\lambda) : H^{-1/2}(\Sigma) \to H^{-1/2}(\Sigma).$$

Theorem (Behrndt, H. 2017)

(i) $\operatorname{dom} \overline{A_{\pm 2c}} \not\subset H^1(\mathbb{R}^3 \setminus \Sigma)$;

(ii) $(-\infty, -mc^2] \cup [mc^2, \infty) \subset \sigma(\overline{A_{\pm 2c}})$;

(iii) *For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ it holds*

$$\left(\overline{A_{\pm 2c}} - \lambda\right)^{-1} = (A_0 - \lambda)^{-1} - \widetilde{\gamma}(\lambda)\left(I + \eta \widetilde{M}(\lambda)\right)^{-1} \eta \gamma(\overline{\lambda})^*.$$

Summary and outlook

Summary:

- Quasi boundary triples are a suitable tool to investigate Dirac operators with singular interactions
- Dirac operators with electrostatic δ-shell interactions of strength $\eta \neq \pm 2c$:
  - Self-adjointness and resolvent formula
  - $\sigma_d(A_\eta)$ is finite
  - The nonrelativistic limit is $-\frac{1}{2m}\Delta + \eta\delta_\Sigma$
- Self-adjointness and resolvent formula for Dirac operators with electrostatic δ-shell interactions of strength $\eta = \pm 2c$

Outlook:

- Spectral properties of $\overline{A_{\pm 2c}}$ in the gap
- More general interaction strengths

Thank you for your attention!