The relativistic δ-shell interaction in $\mathbb{R}^3$ and its approximation by short range potentials

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The setting and the main result

Goal

To couple the Dirac operator with *electrostatic* and *Lorentz scalar potentials* shrinking on the boundary of a smooth domain and to take the limit.

Free Dirac operator in $\mathbb{R}^3$

$H = -i\alpha \cdot \nabla + m\beta$, mass $= m > 0$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_j, \beta \in M_{4\times 4}(\mathbb{C})$,
$\alpha_j^2 = \beta^2 = 1$, $\{\alpha_j, \alpha_k\} = \{\alpha_j, \beta\} = 0 \rightsquigarrow$ Clifford algebra structure.

Remarks

$H : \mathcal{C}_c^\infty(\mathbb{R}^3)^4 \to \mathcal{C}_c^\infty(\mathbb{R}^3)^4$ 1st order symmetric differential operator.
$H^2 = (-\Delta + m^2)I_4 \rightsquigarrow$ Local version of $\sqrt{-\Delta + m^2}$, in the spirit of $4\Delta = \partial_z \partial_{\bar{z}}$ in $\mathbb{R}^2$.
Introduced by Dirac (1928) to study the electron from a relativistic point of view.

Layers and shrinking potentials

$\Omega \subset \mathbb{R}^3$ bounded smooth domain:
$\Sigma = \partial\Omega$, $\sigma =$ surface measure on $\partial\Omega$, $\nu =$ outward normal vector on $\partial\Omega$.
Set $\Sigma_t = \{x_\Sigma + t\nu(x_\Sigma) : x_\Sigma \in \Sigma\} \Longrightarrow \bigcup_{0\le t<\eta} \Sigma_t = \{x \in \mathbb{R}^3 : \operatorname{dist}(x, \Sigma) < \eta\}$.
Given $\eta > 0$ small, $V \in L^\infty(\mathbb{R})$ with $\operatorname{supp} V \subset [-\eta, \eta]$ and $0 < \epsilon \le \eta$, define

$$V_\epsilon(x) = \frac{\eta}{\epsilon} V\left(\frac{\eta t}{\epsilon}\right) \quad \text{for } x = x_\Sigma + t\nu(x_\Sigma) \in \mathbb{R}^3.$$

The setting and the main result

Electrostatic short range and δ-shell interactions

V_ϵ *short range potential:* $\boldsymbol{H + V_\epsilon}$ **is self-adjoint** on the Sobolev space $H^1(\mathbb{R}^3)^4$.

δ*-shell potential:* $\varphi \in H^1(\mathbb{R}^3 \setminus \Sigma)^4$, $\varphi_\pm = \varphi|_\Sigma$ when we approach Σ from Ω or $\mathbb{R}^3 \setminus \overline{\Omega}$.

Set $\boldsymbol{\delta_\Sigma(\varphi) = \frac{1}{2}(\varphi_+ + \varphi_-)\sigma}$. Given $\lambda \in \mathbb{R} \setminus \{\pm 2\}$,

$\boldsymbol{H + \lambda\delta_\Sigma}$ **is self-adjoint** on $\{\varphi \in H^1(\mathbb{R}^3 \setminus \Sigma) : (\alpha \cdot \nu)(\varphi_- - \varphi_+) = \frac{\lambda}{2i}(\varphi_+ + \varphi_-)\}$.

MAIN RESULT

Assume $\operatorname{supp} V \subset [-\eta, \eta]$. Set $u(t) = |\eta V(\eta t)|^{1/2}$, $v(t) = \operatorname{sign}(V(\eta t))u(t)$,

$$\mathcal{K}_V f(t) = \frac{i}{2} \int_{\mathbb{R}} u(t) \operatorname{sign}(t-s) v(s) f(s)\, ds, \;\; f \in L^1_{loc}(\mathbb{R}).$$

Theorem *[M., Pizzichillo, 2016]*

There exist η, $\delta > 0$ small enough such that, for any $\|V\|_{L^\infty(\mathbb{R})} \le \delta/\eta$,

$$\boldsymbol{H + V_\epsilon \to H + \lambda_e \delta_\Sigma} \quad \text{and} \quad \boldsymbol{H + \beta V_\epsilon \to H + \lambda_s \beta\, \delta_\Sigma}$$

in the **strong resolvent sense** when $\epsilon \to 0$, where

$\lambda_e = \int_{\mathbb{R}} v(t)\,((1 - \mathcal{K}_V^2)^{-1} u)(t)\, dt$, $\lambda_s = \int_{\mathbb{R}} v(t)\,((1 + \mathcal{K}_V^2)^{-1} u)(t)\, dt$.

Remarks

[Behrndt, Exner, Holzmann, Lotoreichik, 2015]: $-\Delta + V_\epsilon \to -\Delta + (\int_{\mathbb{R}} V)\delta_\Sigma$.

[Šeba, 1989]: 1D case, same formulas for λ_e, λ_s.

Take $V = \tau\chi_{(-\eta,\eta)}$ for some $\tau \in \mathbb{R}$ such that $0 < |\tau|\eta \le \delta$. Then,

$$H + V_\epsilon \to H + \mathbf{2\tan(\tau\eta)}\delta_\Sigma \quad \text{and} \quad H + \beta V_\epsilon \to H + \mathbf{2\tanh(\tau\eta)}\beta\, \delta_\Sigma.$$

About the proof

Aim

Given $a \in \mathbb{C} \setminus \mathbb{R}$, to show that $(H + V_\epsilon - a)^{-1} \to (H + \lambda_e \delta_\Sigma - a)^{-1}$ in the strong sense in $L^2(\mathbb{R}^3)^4$ when $\epsilon \to 0$.

Main ingredients

(a) **Decompose** $(H + V_\epsilon - a)^{-1}$ using a scaling operator.
(b) Compute the **pointwise limit of each part** when acting on smooth functions.
(c) Show convergence almost everywhere for functions in L^2, and then the strong convergence, using **maximal operators** from Calderón-Zygmund theory.

Step (a)

Scaling operator: $\mathcal{S}_\epsilon g(x_\Sigma, t) = \frac{1}{\sqrt{\epsilon}} g\left(x_\Sigma, \frac{t}{\epsilon}\right)$, $g \in L^2(\Sigma \times (-1,1))^4$.

Decomposition: $(H + V_\epsilon - a)^{-1} = (H - a)^{-1} + A_{\epsilon,a}\big(1 + B_{\epsilon,a}\big)^{-1} C_{\epsilon,a}$,
where $A_{\epsilon,a}, B_{\epsilon,a}, C_{\epsilon,a}$ are defined on $\Sigma \times (-1,1)$
using a fundamental solution of $H - a$, that is,

$$\phi^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}\left(a + m\beta + \left(1 + \sqrt{m^2 - a^2}|x|\right) i\alpha \cdot \frac{x}{|x|^2}\right) \rightsquigarrow \frac{i}{4\pi}\,\alpha \cdot \frac{x}{|x|^3}.$$

Lower order terms: handled as in $-\Delta + V_\epsilon$ (fundamental solution $\sim |x|^{-1}$).

Leading term: in the limit it yields a **singular integral** on Σ. Difficulties to show norm convergence. We require smallness on V to show invertibility of $1 + B_{\epsilon,a}$.

About the proof

Step (b)

We want to compute the limit of $A_{\epsilon,a}, B_{\epsilon,a}, C_{\epsilon,a}$ when $\epsilon \to 0$. We focus on $B_{\epsilon,a}$.
For $(x_\Sigma, t), (y_\Sigma, s) \in \Sigma \times (-1,1)$ set $\boldsymbol{x_{\epsilon t} = x_\Sigma + \epsilon t \nu(x_\Sigma)}$, $\boldsymbol{y_{\epsilon s} = y_\Sigma + \epsilon s \nu(y_\Sigma)}$.
Then,

$$\boldsymbol{B_{\epsilon,a}} f(x_\Sigma, t) = u(t) \int_{-1}^{1} \int_{\Sigma_{\epsilon s}} \boldsymbol{\phi^a(x_{\epsilon t} - y_{\epsilon s})} v(s) f(y_\Sigma, s)\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds.$$

The leading term of $\phi^a(x) \rightsquigarrow \boldsymbol{k(x) = \frac{x}{4\pi|x|^3}}$. Set

$$\boldsymbol{T_\epsilon} f(x_\Sigma, t) = \int_{-1}^{1} \int_{\Sigma_{\epsilon s}} \boldsymbol{k(x_{\epsilon t} - y_{\epsilon s})} f(y_\Sigma, s)\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds,$$

$$\boldsymbol{T} f(x_\Sigma, t) = \lim_{\delta \to 0} \int_{-1}^{1} \int_{|x_\Sigma - y_\Sigma| > \delta} k(x_\Sigma - y_\Sigma) f(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds + \frac{\nu(x_\Sigma)}{2} \int_{-1}^{1} \operatorname{sign}(t-s) f(x_\Sigma, s)\, ds.$$

We want to show that $\boldsymbol{T_\epsilon f(x_\Sigma, t) \to Tf(x_\Sigma, t)}$ when $\epsilon \to 0$.

Split $k = (k_1, k_2, k_3)$ in *normal* and *tangential* components:

$$4\pi k_j(x-y) = \boldsymbol{\frac{x_j - y_j}{|x-y|^3} = \left(\frac{x-y}{|x-y|^3} \cdot \nu(y)\right) \nu_j(y) + \tan_{x,y,j}}.$$

About the proof – continuation of **Step (b)**

Recall that $T_\epsilon f(x_\Sigma, t) = \int_{-1}^{1}\int_{\Sigma_{\epsilon s}} k(x_{\epsilon t} - y_{\epsilon s}) f(y_\Sigma, s)\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds$.
We focus on the *normal* component:

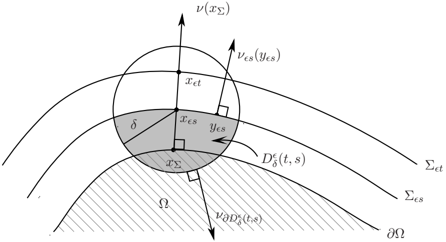
$$\frac{x_j - y_j}{|x-y|^3} \rightsquigarrow \Big(\frac{x-y}{|x-y|^3}\cdot \nu(y)\Big)\nu_j(y) \quad \Longrightarrow \quad \boldsymbol{T_\epsilon \rightsquigarrow T_\epsilon^\nu}, \quad \boldsymbol{f \rightsquigarrow \nu_j f = f_j}.$$

Assume that $\boldsymbol{f_j}$ is smooth on $\boldsymbol{\Sigma \times (-1,1)}$. Then, given $\delta > 0$ we can split

$$\begin{aligned} \boldsymbol{T_\epsilon^\nu} f_j(x_\Sigma, t) = &\int_{-1}^{1}\int_{|x_{\epsilon s}-y_{\epsilon s}|>\delta} \boldsymbol{k(x_{\epsilon t} - y_{\epsilon s})}\cdot \nu_{\epsilon s}(y_{\epsilon s})\, f_j(y_\Sigma, s)\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds \\ &+ \int_{-1}^{1}\int_{|x_{\epsilon s}-y_{\epsilon s}|\leq\delta} k(x_{\epsilon t} - y_{\epsilon s})\cdot \nu_{\epsilon s}(y_{\epsilon s})\Big(\boldsymbol{f_j(y_\Sigma, s) - f_j(x_\Sigma, s)}\Big)\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds \\ &+ \int_{-1}^{1} f_j(x_\Sigma, s)\int_{|x_{\epsilon s}-y_{\epsilon s}|\leq\delta} \boldsymbol{k(x_{\epsilon t} - y_{\epsilon s})\cdot \nu_{\epsilon s}(y_{\epsilon s})}\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds. \end{aligned}$$

$$\boldsymbol{D_\delta^\epsilon(t,s)} = \begin{cases} B_\delta(x_{\epsilon s}) \setminus \overline{\Omega(\epsilon, s)} & \text{if } t \leq s, \\ B_\delta(x_{\epsilon s}) \cap \Omega(\epsilon, s) & \text{if } t > s, \end{cases}$$

where $\Omega(\epsilon, s)$ is the bounded connected component of $\mathbb{R}^3 \setminus \Sigma_{\epsilon s}$ that contains Ω if $s \geq 0$ and that is included in Ω if $s < 0$.



$D_\delta^\epsilon(t,s)$ in the case $t > s > 0$.

About the proof

Step (c)

Hardy-Littlewood maximal operator: $\boldsymbol{M_* g(x_\Sigma)} = \sup_{\delta>0} \frac{1}{\sigma(B_\delta(x_\Sigma))} \int_{B_\delta(x_\Sigma)} |g|\, d\sigma.$

Maximal SIO: $\boldsymbol{T_* g(x_\Sigma)} = \sup_{\delta>0} \left| \int_{|x_\Sigma - y_\Sigma|>\delta} k(x_\Sigma - y_\Sigma) g(y_\Sigma)\, d\sigma(y_\Sigma) \right|.$

Covering lemmas and Calderón-Zygmund theory (Σ is smooth, σ Ahlfors regular)
$\Longrightarrow$ **M_* and T_* are bounded in L^2**.

Estimates on k: $|k(x)| \lesssim \frac{1}{|x|^2}$, $|k(z-y) - k(x-y)| \lesssim \frac{|z-x|}{|x-y|^3}$ if $|z-x| \leq \frac{1}{2}|x-y|$.

Recall that $T_\epsilon f(x_\Sigma, t) = \int_{-1}^{1} \int_{\Sigma_{\epsilon s}} k(x_{\epsilon t} - y_{\epsilon s}) f(y_\Sigma, s)\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds.$
Since $\boldsymbol{f}(y_\Sigma, s)\, d\boldsymbol{\sigma_{\epsilon s}}(y_{\epsilon s}) \sim \boldsymbol{f_\epsilon}(y_\Sigma, s)\, d\boldsymbol{\sigma}(y_\Sigma)$ (use the Weingarten map), we decompose

$$\begin{aligned} \boldsymbol{T_\epsilon} f(x_\Sigma, t) = & \int_{-1}^{1} \int_{\boldsymbol{|x_\Sigma - y_\Sigma| \leq 4\epsilon|t-s|}} k(x_{\epsilon t} - y_{\epsilon s}) f_\epsilon(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds \\ & + \int_{-1}^{1} \int_{|x_\Sigma - y_\Sigma| > 4\epsilon|t-s|} \big(k(\boldsymbol{x_{\epsilon t}} - y_{\epsilon s}) - k(\boldsymbol{x_{\epsilon s}} - y_{\epsilon s})\big) f_\epsilon(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds \\ & + \int_{-1}^{1} \int_{|x_\Sigma - y_\Sigma| > 4\epsilon|t-s|} \big(k(\boldsymbol{x_{\epsilon s} - y_{\epsilon s}}) - k(\boldsymbol{x_\Sigma - y_\Sigma})\big) f_\epsilon(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds \\ & + \int_{-1}^{1} \int_{\boldsymbol{|x_\Sigma - y_\Sigma| > 4\epsilon|t-s|}} k(x_\Sigma - y_\Sigma) f_\epsilon(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds. \end{aligned}$$

$\Longrightarrow$ $\boldsymbol{|T_\epsilon f(x_\Sigma, t)| \lesssim \mathcal{M}_* f(x_\Sigma) + \mathcal{T}_* f(x_\Sigma)}$, variants of M_* and T_* on $\Sigma \times (-1, 1)$.

About the proof – continuation of **Step (c)**

We have seen that $|T_\epsilon f(x_\Sigma, t)| \lesssim \mathcal{M}_* f(x_\Sigma) + \mathcal{T}_* f(x_\Sigma)$ for all $\epsilon > 0$ and $f \in L^2$,
$\mathcal{M}_*$ and $\mathcal{T}_*$ are bounded in L^2.
From **Step (b)**, if **f is smooth** on $\Sigma \times (-1, 1)$, $T_\epsilon f(x_\Sigma, t) \to Tf(x_\Sigma, t)$ when $\epsilon \to 0$.

Given $\lambda > 0$ and $f \in L^2$, take $f_k \to f$, f_k smooth.

$$\Big|\Big\{(x_\Sigma, t) \in \Sigma \times (-1, 1) : \Big|\limsup_{\epsilon \to 0} T_\epsilon f(x_\Sigma, t) - \liminf_{\epsilon \to 0} T_\epsilon f(x_\Sigma, t)\Big| > \lambda\Big\}\Big|$$
$$\leq \Big|\Big\{\Big|\limsup_{\epsilon \to 0} T_\epsilon (f - f_k)(x_\Sigma, t)\Big| + \Big|\liminf_{\epsilon \to 0} T_\epsilon (f - f_k)(x_\Sigma, t)\Big| > \lambda\Big\}\Big|$$
$$\leq \big|\{\mathcal{M}_*(f - f_k)(x_\Sigma) + \mathcal{T}_*(f - f_k)(x_\Sigma) > C\lambda\}\big| \leq \frac{C}{\lambda^2} \|f - f_k\|_2^2.$$

$\Longrightarrow$ **$T_\epsilon f(x_\Sigma, t) \to Tf(x_\Sigma, t)$ almost everywhere** when $\epsilon \to 0$, for all $f \in L^2$.

Maximal estimates + *Dominated convergence* $\Longrightarrow$ **$T_\epsilon f \to Tf$ in L^2**.
Thus $T_\epsilon \to T$ in the strong sense when $\epsilon \to 0$.

Thanks for your attention.