Quantization of the Maxwell-Dirac equations

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Action functional

$$S = \int \Big[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\Psi} \Big(i\hbar c \not{D}_A - \frac{mc^2}{\hbar} \Big) \Psi \Big] \, dx \, dt$$

describing the interaction of a Dirac spinor field Ψ with an electromagnetic field F in Minkowski space-time with coordinates $(x^0 = ct, \mathbf{x})$ and metric $c^2 dt^2 - d\mathbf{x}^2$. F is given in terms of the potential A 1-form by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and

$$\boxed{\not{D}_A = \gamma^\mu (\partial_\mu - \frac{ie}{\hbar c} A_\mu)}$$

where $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, with g the (here Minkowski) metric. Write $\overline{\Psi} = \Psi^\dagger \gamma^0$, where $\dagger$ means Hermitian conjugate. The equations of motion are

$$\partial_\mu F^{\mu\nu} = -e \, \overline{\Psi} \gamma^\nu \Psi \, ,$$
$$i \not{D}_A \Psi = \frac{mc}{\hbar} \Psi \, .$$

Gauge invariance: $\psi \to e^{ig} \psi$ and $a_\mu \to a_\mu + \partial_\mu g$ where $g = g(t, x)$ is a sufficiently regular function.

$$S = \int \Big[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\Psi} \Big(i\hbar c \not{D}_A - \frac{mc^2}{\hbar} \Big) \Psi \Big] \, dx \, dt$$

Classical equations: 3+1-D well posed for $\Psi(0) \in H^s$, $F(0) \in H^{s-\frac{1}{2}}$ for $s > 0$, unique with $X^{s,b}$ condition. [d'Ancona Foschi Selberg]

Existence of quantum theory in 3+1-D in doubt.

Classical equations well posed for $s = 0$ "charge class" in 1+1-D and 2+1-D.

Quantum theory in 1+1-D. Compare to classical

$$\psi = \frac{1}{\sqrt{L}} \sum_{n \in \mathbb{Z}} \Big(b_n u_n e^{ik_n x} + c_n^\dagger v_n e^{-ik_n x} \Big) , \quad k_n = \frac{2n\pi}{L}$$

Classically: plane waves. Regularity $\leftrightarrow$ decay of b_n, c_n

Quantum Heisenberg anti-CR:

$$\boxed{\{b_n, b_m^\dagger\} = \mathbb{1} \leftrightarrow \{\psi(x), \psi(y)^\dagger\} = \mathbb{1}\delta(x - y)}$$

- operators $\{b_n \dots\}$ are bounded, no decay.

- Operator-valued distribution $\psi(x)$ irregular.

Test function $f(x) = \sum \widehat{f}(n) e^{ik_n x}$, then $\psi(f)$ bounded operator if $\sum |\widehat{f}(n)| < \infty$.

Massless case: chiral invariance and potential formulation

Dirac equation coupled to electromagnetic field

$$i\not{D}_A\Psi = \frac{mc}{\hbar}\Psi$$

In massless case

$$\boxed{m = 0}$$

can "γ^5 gauge away" the field A

$$\Psi = [ie(\theta + \gamma^5\chi)]\psi$$

where $\boxed{\not{D}\psi = 0}$ and

$$A_\mu = \partial_\mu\theta - \epsilon_\mu^\nu\partial_\nu\chi$$

with

$$\Box\chi = -f\,,\ \Box\theta = \partial\cdot A\,,\ \Box f = 0$$

Here f is a potential for the current $J_\mu = -e\overline{\Psi}\gamma_\mu\Psi$, i.e. $\partial_\mu J^\mu = 0 \implies J^\mu = \epsilon^{\mu\nu}\partial_\nu f$ for some f. This reduces the whole system to free waves.

II Bound states in the Einstein-Dirac system

Action functional

$$S = \frac{c^3}{8\pi G}\int R\, d\mu_g + \sum_{A=1}^{2} \hbar \int \overline{\Psi}_A\Big(\not{D} - \frac{mc}{\hbar}\Big)\Psi_A\, d\mu_g$$

describing the interaction of two Dirac spinor fields Ψ_1 and Ψ_2 with a gravitational metric g, whose scalar curvature is R and whose volume element is $d\mu_g$. $\not{D} =$ the Dirac operator derived from g, coupled through associated γ matrices. The Euler-Lagrange equations are

$$R_{ab} - \frac{1}{2} R g_{ab} = \frac{8\pi G}{c^4} T_{ab}, \qquad (\not{D} - \frac{mc}{\hbar})\Psi_A = 0$$

where R_{ab} is the Ricci curvature and

$$T_{ab} = \frac{\hbar c}{2} \sum \mathrm{Re}\left[\overline{\Psi}_A (i\gamma_a \partial_b + i\gamma_b \partial_a)\Psi_A\right]$$

is the energy-momentum tensor; space-time indices a, b take values in $\{0, 1, 2, 3\}$.

Finster-Smoller-Yau ansatz for spherical symmetry:

$$g = c^2 e^{2\nu} dt^2 - e^{2\lambda} dr^2 - r^2 d\Omega^2$$

and spinor fields of the form

$$\Psi_1 = e^{\nu/2} e^{-i\omega t} \begin{pmatrix} \Phi_1 e_1 \\ i\Phi_2 \sigma^r e_1 \end{pmatrix}, \quad \Psi_2 = e^{\nu/2} e^{-i\omega t} \begin{pmatrix} \Phi_1 e_2 \\ i\Phi_2 \sigma^r e_2 \end{pmatrix}$$

where

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \sigma^r = \frac{1}{r} \sum_{i=1}^{3} x^i \sigma^i$$

where σ^i are the Pauli matrices.

$$\omega \hbar = mc^2 + \eta$$

The angular dependence of Ψ_1, Ψ_2 is that displayed by the ground state Dirac wave functions for the relativistic The states Ψ_1, Ψ_2 have opposite values of j_3, which ensures that the energy momentum tensor T_{ab} is consistent with the spherically symmetric metric.

The spherically symmetric Einstein-Dirac system admits nonlinear bound state solutions $(\lambda^\epsilon, \nu^\epsilon, \Phi_1^\epsilon, \Phi_2^\epsilon)$ *for small positive* ϵ*, which can be approximated (in a strong weighted norm) by the bound state solution* φ *of the Newton-Schrödinger system which minimizes the energy : in particular,* $(\Phi_1^\epsilon, \Phi_2^\epsilon)$ *converges uniformly to* $(\varphi, 0)$ *as* $\epsilon \to 0$*.*

The Newton-Schrödinger system

Theorem 1 (Lieb)**.** *The associated nonlocal energy*

$$\frac{\hbar^2}{2m}\int |\nabla\varphi(x)|^2 dx - m^2 G \int\int \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|} dx dy$$

admits a finite lower bound subject to the constraint of having $\int |\varphi(x)|^2 dx = 1$ fixed, and this lower bound is attained on a function which is unique up to translation. Further this minimizer is positive, spherically symmetric and a monotone non-increasing function of the radial coordinate satisfying $|\varphi(r)| \leq c_1 e^{-c_2 r}$ for some positive numbers c_1, c_2.

We summarize some points:

1. The existence of a spherically symmetric minimizer of this nonlocal energy is proved by means of the Riesz rearrangement inequality, and a strict version of this inequality implies that any minimizer is spherically symmetric. The corresponding Euler-Lagrange equation is

$$-\frac{\hbar^2}{2m}\Delta\varphi(x) - 2m^2 G \int \frac{|\varphi(y)|^2}{|x-y|} dy\, \varphi(x) = \eta\varphi(x)$$

where $\eta < 0$ is the Lagrange multiplier.

2. The relation between this equation and the energy follows quickly using the condition $\lim_{|x|\to+\infty} |u(x)| = 0$ and the formula for the solution of Poisson's equation $-\Delta u = f$ on $\mathbf{R}^3$, namely:

$$(-\Delta^{-1} f)(x) = \int \frac{f(y)}{4\pi|x-y|} dy.$$

3. For the case when $f(x) = \kappa\rho(r)$, where $\kappa > 0$ is a constant and ρ is a function of the radial coordinate $r = |x|$ only, a result of Newton implies:

$$(-\Delta^{-1}\kappa\rho)(r) = -\kappa\int_0^r\left(\frac{1}{s}-\frac{1}{r}\right)\rho(s)s^2ds + \kappa\int_0^\infty \rho(s)sds$$
$$= -\kappa\int_0^r\left(\frac{1}{s}-\frac{1}{r}\right)\rho(s)s^2ds + u(0).$$

Define $K(r,s) \equiv 8\pi s^2(\frac{1}{s}-\frac{1}{r})$; this kernel is non-negative for $0 \leq s \leq r$. Lieb proved any energy minimizing solution to is radially symmetric, and so solves the equation

$$E\varphi = -\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)\varphi + m^2G\left(\int_0^r K(r,s)|\varphi(s)|^2ds\right)\varphi$$

where $E = (\eta - mu(0))$.

4. Furthermore all positive solutions of this latter equation can be obtained by a scaling of the unique positive solution of

$$-\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)\phi + \left(\int_0^r K(r,s)|\phi(s)|^2ds\right)\phi = \phi.$$

Theorem 2. *There exists an interval* $(-\epsilon_1, +\epsilon_1)$ *on which is defined a* C^1 *curve* $\epsilon \to \Xi^\epsilon = (l^\epsilon, Q^\epsilon, N^\epsilon, \Phi_1^\epsilon, \psi_2^\epsilon) \in X$ *of solutions to the Einstein-Dirac system such that* $\|\Xi^\epsilon - \Xi_N\|_X = O(\epsilon)$. *More explicitly,* $l^\epsilon \to 2mG$ *as* $\epsilon \to 0$ *and*

$$\|Q^\epsilon + 4mGf_0 - 2ru'\|_{BC_2^{1,2}} + \|N^\epsilon - u\|_{BC^{1,1}} + \|\Phi_1^\epsilon - \varphi\|_{H_{ra}^{\{2}}$$

$$+ \|\psi_2^\epsilon + \frac{\hbar}{2m}\varphi'\|_{H_{rad}^{\{1,\delta\}} \cap BC_1^{1,0}} = O(\epsilon).$$

In terms of the original variables of the problem, we define a metric

$$g^\epsilon = \epsilon^{-2} e^{2\nu^\epsilon} dt^2 - e^{2\lambda^\epsilon} dr^2 - r^2 d\Omega^2$$

where $\nu^\epsilon = \epsilon^2 N^\epsilon$ and $(1 - e^{-2\lambda^\epsilon}) = \epsilon^2(2l^\epsilon f_0(r) + Q^\epsilon)$. Define also $\Phi_2^\epsilon = \epsilon\psi_2^\epsilon$, then for $\epsilon = c^{-1}$ small we have a solution $(\lambda^\epsilon, \nu^\epsilon, \Phi_1^\epsilon, \Phi_2^\epsilon)$ and

$$\epsilon^{-2}\|(1 - e^{-2\lambda^\epsilon}) - 2\epsilon^2 ru'\|_{BC_2^{1,2}} + \epsilon^{-2}\|\nu^\epsilon - \epsilon^2 u\|_{BC^{1,1}}$$

$$+ \|\Phi_1^\epsilon - \varphi\|_{H_{rad}^{\{2,\delta\}} \cap BC^{1,0}} + \epsilon^{-1}\|\Phi_2^\epsilon + \epsilon\frac{\hbar}{2m}\varphi'\|_{H_{rad}^{\{1,\delta\}} \cap B}$$

$$= O(\epsilon).$$

- In addition to results on Newton-Schrodinger bound states proof depends on their non-degeneracy (Lenzmann).
- Weights in norms

$$\|f\|_{BC^{\delta}_{\delta'}} = \sup r^{-\delta'}|f(r)| + \sup(1+r)^{\delta}|f(r)|$$

etc have to be chosen to allow dynamical adjustment of Arnowitt-Deser-Misner mass.
- J. Math. Phys. 51 (2010), 032501 and also Rota-Nodari, Ann IHP (2010).
- Existence of such solutions reflects attractive nature of gravity. Why do such solutions exist for Maxwell-Dirac system?

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Stationary solutions of the Maxwell-Dirac and the Klein-Gordon-Dirac equations

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Abstract. The Maxwell-Dirac system describes the interaction of an electron with its own electromagnetic field. We prove the existence of soliton-like solutions of Maxwell-Dirac in (3+1)-Minkowski space-time. The solutions obtained are regular, stationary in time, and localized in space. They are found by a variational method, as critical points of an energy functional. This functional is strongly indefinite and presents a lack of compactness. We also find soliton-like solutions for the Klein-Gordon-Dirac system, arising in the Yukawa model.

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1. Introduction

The Maxwell-Dirac equations, which describe the interaction of an electron with its own electromagnetic field, play a major role in quantum electrodynamics. They can be written as follows

$$(M-D)\qquad \begin{cases} (i\gamma^\mu\partial_\mu - \gamma^\mu A_\mu)\psi - m\psi = 0 & \text{in } \mathbb{R}\times\mathbb{R}^3 \\ \partial_\mu A^\mu = 0 \quad , \quad 4\pi\,\partial_\mu\partial^\mu A^\nu = J^\nu & \text{in } \mathbb{R}\times\mathbb{R}^3 \end{cases}$$

where $\nu,\mu \in \{0,1,2,3\}$, $m>0$, $(\ ,\ )$ is the usual hermitian product in $\mathbb{C}^4$, $\psi(x_0,x)\in\mathbb{C}^4$ for $(x_0,x)\in\mathbb{R}\times\mathbb{R}^3$ and $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix} \in M_{4\times4}(\mathbb{C})$, $\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0\end{pmatrix} \in \mathscr{M}_{4\times4}(\mathbb{C})$, $\overline{\psi} = \gamma^0\psi$, $J^\mu = (\overline{\psi},\gamma^\mu\psi)$, $J_0 = J^0$, $J_k = -J^k$, $k=1,2,3$, and σ^k are the Pauli matrices

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$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Solutions of (M-D) that are stationary in time, and localized in space, are called soliton-like solutions of Maxwell-Dirac. They can be viewed as representations of extended particles. Their existence has been an open problem for a long time (see e.g. [18], p.235). It is the aim of this paper to find such solutions. We also find soliton-like solutions for the Klein-Gordon-Dirac equations arising in the so-called Yukawa model (see [7] and [4]). These equations are

$$(KG-D) \qquad \begin{cases} (i\gamma^\mu\partial_\mu - \chi)\psi - m\psi = 0 & \text{in } \mathbb{R}\times\mathbb{R}^3 \\ \partial^\mu\partial_\mu\chi + M^2\chi = \dfrac{1}{4\pi}(\overline{\psi},\psi) & \text{in } \mathbb{R}\times\mathbb{R}^3 . \end{cases}$$

The above systems have been studied for a long time and many results are available concerning the Cauchy problem for (M-D). The first result about the local existence and uniqueness of solutions of (M-D) was obtained by L. Gross in [19]. Later developments were made by Chadam [10] and Chadam and Glassey [11] in $1+1$ and $2+1$ space-time dimensions and in $3+1$ dimensions when the magnetic field is 0. Choquet-Bruhat studied in [12] the case of spinor fields of zero mass and Maxwell-Dirac equations in the Minskowski space were studied by Flato, Simon and Taflin in [15]. In [17], Georgiev obtained a class of initial values for which the Maxwell-Dirac equations have a global solution. This was performed by using a technique of Klainerman (see [21-25]), which gives L^∞ a priori estimates via the Lorentz invariance of the equations, and a generalized version of the energy inequalities. In this respect, see also [21]. The same method was used by Bachelot [1] to obtain a similar result for (KG-D). Finally, recent results of Beals and Bezard yield the existence and uniqueness of weak solutions for initial data satisfying the natural energy estimates.

As far as the existence of stationary solutions (soliton-like) of (M-D) is concerned, there is a pionneering work by Wakano ([32]) in which an approximation of (M-D) is studied:

Assuming that the electrostatic potential is predominant, the extreme case in which $A_0 \not\equiv 0, A_1 = A_2 = A_3 \equiv 0$ is considered (Coulomb-Dirac). The approximate problem (C-D) can be reduced to a system of three coupled differential equations by using the spherical spinors. Wakano obtained numerical evidence for the existence of stationary solutions of (C-D). Further work in this direction (see [28, 30]) yielded the same kind of numerical results for some modified Maxwell-Dirac equations which include some nonlinear self-coupling.

Recently, Garret Lisi (see [16]) obtained numerical solutions for the whole system of Maxwell-Dirac equations. This was done by using an axially symmetric ansatz.

In the present paper we make no approximation on the electromagnetic potential, and we show that for $0 < \omega < m$ there are exact solutions

$(\psi, A) : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}^4 \times \mathbb{R}^4$ of (M-D) of the form

$$
(1.1) \qquad \begin{cases} \psi(x_0, x) = e^{i\omega x_0}\varphi(x) \quad , \quad \varphi : \mathbb{R}^3 \to \mathbb{C}^4 \\ A^\mu(x_0, x) = J^\mu * \dfrac{1}{|x|} = \displaystyle\int_{\mathbb{R}^3} \frac{dy}{|x-y|}\, J^\mu(y) \, . \end{cases}
$$

We prove this result by using a variational method which was introduced by Esteban and Séré in [14] (see also [13]) to deal with some class of nonlinear Dirac equations in which the nonlinear coupling is local, the so-called Soler model (for more details and results on this model, see e.g. [2, 3, 8, 9, 27, 28, 30]). This variational method was inspired by earlier works on periodic and homoclinic orbits of hamiltonian systems ([6, 5, 20, 31, 29]).

In order to state the main results contained in this paper, let us note that

If (ψ, A) is a solution of (M-D) of the form (1.1), then (φ, A) is a solution of

$$
(1.2) \qquad \begin{cases} i\gamma^k \partial_k \varphi - m\varphi - \omega\gamma^0\varphi - \gamma^\mu A_\mu \varphi = 0 & \text{in } \mathbb{R}^3 \\ -4\pi \Delta A_0 = J^0 = |\varphi|^2 \quad , \quad -4\pi\Delta A_k = -J^k & \text{in } \mathbb{R}^3 \, . \end{cases}
$$

The solutions of (1.2) are given by the critical points $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$ of the functional

$$
(1.3) \qquad \begin{aligned} I_\omega(\varphi) = & \int_{\mathbb{R}^3} \frac{1}{2}(i\gamma^0\gamma^k\partial_k\varphi, \varphi) - \frac{m}{2}(\overline{\varphi}, \varphi) - \frac{\omega}{2}|\varphi|^2 \\ & - \frac{1}{4}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{J^\mu(x) J_\mu(y)}{|x-y|}\, dx\, dy \, . \end{aligned}
$$

Our main result concerning the Maxwell-Dirac equations is the following.

Theorem 1. *For any $\omega \in (0, m)$ there exists a non-zero critical point φ_ω of I_ω in $H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$. φ_ω is a smooth function of x, exponentially decreasing at infinity together with all its derivatives. The fields $\psi(x_0, x) = e^{i\omega x_0}\varphi_\omega$, $A^\mu(x_0, x) = J^\mu_\omega * \frac{1}{|x|}$ are solutions of the Maxwell-Dirac system* (M-D).

The stationary solutions of the Klein-Gordon-Dirac equations are given by critical points $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$ of the functional

$$
(1.4) \qquad \begin{aligned} J_\omega(\varphi) = & \int_{\mathbb{R}^3} \frac{1}{2}(i\gamma^0\gamma^k\partial_k\varphi, \varphi) - \frac{m}{2}(\overline{\varphi}, \varphi) - \frac{\omega}{2}|\varphi|^2 \\ & - \frac{1}{4}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(\overline{\varphi}, \varphi)(x)\,(\overline{\varphi}, \varphi)(y)}{|x-y|}\, e^{-M|x-y|}\, dx\, dy \, . \end{aligned}
$$

About this problem we will prove the following.

Theorem 2. *There are infinitely many critical points of J_ω for any $\omega \in (0, m)$. These critical points have the form*

$$
(1.5) \qquad \varphi(x) = \begin{pmatrix} v(r) & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ iu(r) & \begin{pmatrix} \cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix} \end{pmatrix} ,
$$

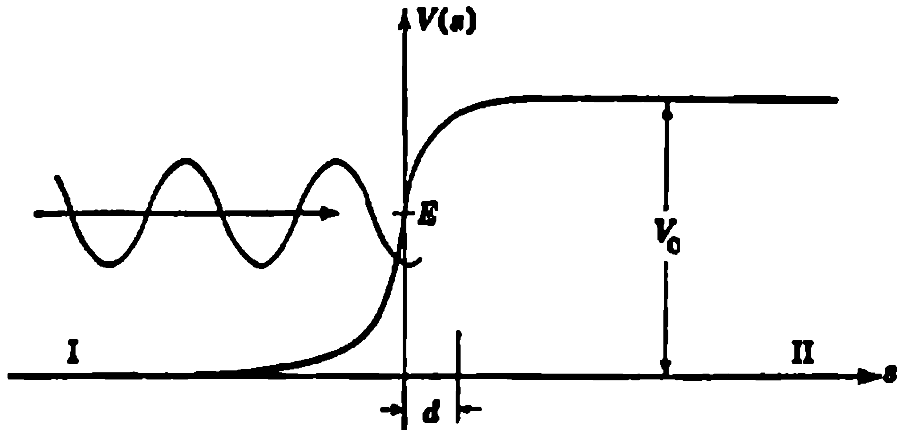

Fig. 3-1 Potential barrier confining electron of energy E in region I to the left.

This result can be equally well inferred on dimensional groups using $\Delta p\,\Delta x \sim \hbar$ without reference to the particular gaussian shape. In discussing problems and interactions in which the electron is "spread out" over distances large compared with its Compton wavelength, we may simply ignore the existence of the uninterpreted negative-energy solutions and hope to obtain physically sensible and accurate results. This will not work, however, in situations which find electrons localized to distances comparable with $\hbar/mc$. The negative-frequency amplitudes will then be appreciable, the zitterbewegung terms in the current important, and indeed we shall find ourselves beset by paradoxes and dilemmas which defy interpretation within the framework so far developed by the Dirac theory of an electron. A celebrated example of these difficulties is the Klein paradox,[1] illustrated by the following example.

In order to localize electrons, we must introduce strong external forces confining them to the desired region. Suppose, for example, we want to confine a free electron of energy E to region I to the left of the origin $z = 0$ in the one-dimensional potential diagram of Fig. 3.1. If the electron is not to be found more than a distance d to the right of $z = 0$, in region II, then V must rise sharply within an interval $z \lesssim d$ to a height $V_0 > E$ so that the solution in II falls off with a characteristic width $\lesssim d$. This is as in the Schrödinger theory, until the confining length d shrinks to $\sim\hbar/mc$ and $V_0 - E$ increases beyond mc^2. To see what happens, let us consider an electrostatic potential with a sharp boundary as in Fig. 3.2 and calculate the reflected and transmitted current for an electron of wave number k incident from the left with spin up along the z direction. The positive-energy solutions for the incident and reflected waves in region I may be

[1] O. Klein, *Z. Physik*, **53**, 157 (1929).

written

$$\psi_{\text{inc}} = ae^{ik_1z}\begin{bmatrix} 1 \\ 0 \\ \dfrac{ck_1\hbar}{E+mc^2} \\ 0 \end{bmatrix} \tag{3.34}$$

$$\psi_{\text{ref}} = be^{-ik_1z}\begin{bmatrix} 1 \\ 0 \\ -\dfrac{ck_1\hbar}{E+mc^2} \\ 0 \end{bmatrix} + b'e^{-ik_1z}\begin{bmatrix} 0 \\ 1 \\ 0 \\ \dfrac{ck_1\hbar}{E+mc^2} \end{bmatrix}$$

For the transmitted wave we need the solutions of the Dirac equation in the presence of a constant external potential $e\Phi = V_0$. These differ from the free-particle solutions only by the substitution $p_0 = (1/c)(E - V_0)$, so that in region II

$$\hbar^2k_2^2c^2 = (E - V_0)^2 - m^2c^4 = (E - mc^2 - V_0)(E + mc^2 - V_0)$$

We therefore write the transmitted wave of positive energy $E > 0$ as

$$\psi_{\text{trans}} = de^{ik_2z}\begin{bmatrix} 1 \\ 0 \\ \dfrac{c\hbar k_2}{E - V_0 + mc^2} \\ 0 \end{bmatrix} + d'\,e^{ik_2z}\begin{bmatrix} 0 \\ 1 \\ 0 \\ \dfrac{-c\hbar k_2}{E - V_0 + mc^2} \end{bmatrix} \tag{3.35}$$

The amplitudes d and d' are fixed by continuity of the solution at

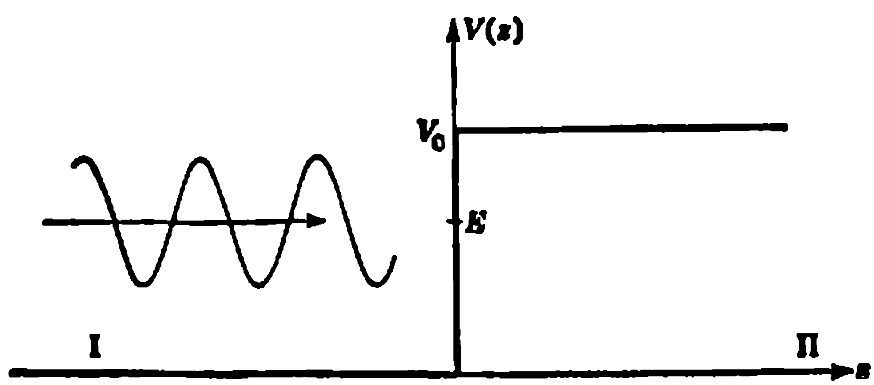

Fig. 3-2 Electrostatic potential idealized with a sharp boundary, with an incident free electron wave of energy E moving to the right in region I. For $V_0 > E + mc^2$ the reflected current from the potential exceeds the incident one; this is an example of the Klein paradox.

“Paradoxical” aspects of Dirac equation

The single particle Dirac equation describes relativistic particles of (negative) charge.

- Repulsive potential can produce bound states from negative energy spectrum
- Klein paradox: Propagation through large potential barriers of height $V_0 > E$ where transmission by Schrodinger’s equation disallowed. Continuous spectrum of free Dirac operator is

  $$(-\infty, -mc^2] \cup [mc^2, +\infty)$$

  Electron with energy $mc^2 < E < -mc^2 + V_0 < V_0$ can tunnel through barrier.
  Should be explained though multi particle theory - see discussion in of Bongaarts and Ruijsenaars (1976) in context of $C*$ algebraic quantization of Dirac equation in external potential. Pessimistic conclusion on possibility of unitary scattering operator.

Repulsion and formation of bound states

Work in $1+1$ dimensional Minkowski space-time with coordinates $(x^0 = ct, x^1 = x)$ and metric $c^2dt^2 - dx^2$. Can see phenomenon very clearly.

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{1}$$

Electromagnetic field consists only

$$E = F_{01} = \frac{1}{c}\dot{A}_1 - \partial_x A_0$$

$$\boldsymbol{\alpha} = \gamma^0\gamma^1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \boldsymbol{\beta} = \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

and using the gauge in which A_1 is zero these equations reduce to

$$-A_0'' = \qquad -e\,\overline{\Psi}\boldsymbol{\beta}\Psi \tag{3}$$

$$i\dot{\Psi} + ic\boldsymbol{\alpha}\Psi' = \qquad +\frac{mc^2}{\hbar}\boldsymbol{\beta}\Psi - \frac{e}{\hbar}A_0\Psi\,. \tag{4}$$

Consider bound state solutions of form:

$$\Psi(t,x) = e^{-iEt/\hbar}\begin{pmatrix} U(x) \\ V(x) \end{pmatrix}, \qquad A_0(t,x) = -\varphi(x)\,. \tag{5}$$

Repulsion and formation of bound states II

$$-\varphi'' = e\Big(|U|^2 + |V|^2\Big)\,,$$
$$\hbar c V' = (E - e\varphi - mc^2)U\,,$$
$$\hbar c U' = -(E - e\varphi + mc^2)V\,.$$

Put $E = -mc^2 + \eta$ and observe that formally $U/V = O(\frac{1}{c}) = O(\epsilon)$. Rescale $U = \epsilon\tilde{U}$

$$-\varphi'' = e\Big(\epsilon^2|\tilde{U}|^2 + |V|^2\Big)\,,$$
$$\hbar V' + (2m - \epsilon^2\eta)\tilde{U} = -\,e\,\varphi\,\epsilon^2\tilde{U}\,,$$
$$\hbar\tilde{U}' + \eta\,V = e\,\varphi\,V\,.$$

This can be treated as perturbation of

$$-\frac{\hbar^2}{2m} - e\,\varphi\,V = -\,\eta\,V$$

Now have an attractive potential for Schrodinger equation.

I Maxwell Dirac equations

The starting point is the action functional

$$S = \int \Big[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\Psi} \Big(i\hbar c \not{D}_A - \frac{mc^2}{\hbar} \Big) \Psi \Big] \, dx \, dt$$

describing the interaction of a Dirac spinor field Ψ with an electromagnetic field in Minkowski space-time with coordinates $(x^0 = ct, \mathbf{x})$ and metric $c^2 dt^2 - d\mathbf{x}^2$. The electromagnetic field is given in terms of the potential A 1-form by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. he Dirac operator is given by

$$\not{D}_A = \gamma^\mu (\partial_\mu - \frac{ie}{\hbar c} A_\mu)$$

where $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, with g the (here Minkowski) metric. Write $\overline{\Psi} = \Psi^\dagger \gamma^0$, where $\dagger$ means Hermitian conjugate. The equations of motion are

$$\partial_\mu F^{\mu\nu} = -e \, \overline{\Psi} \gamma^\nu \Psi \, ,$$
$$i \not{D}_A \Psi = \frac{mc}{\hbar} \Psi \, .$$

Gauge invariance: $\psi \to e^{ig} \psi$ and $a_\mu \to a_\mu + \partial_\mu g$ where $g = g(t, x)$ is a sufficiently regular function.

Dirac γ-matrices are:

$$\gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

where I_2 is the 2×2 unit matrix and σ_j are the Pauli matrices: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. After introduction of a space time splitting:

$$i\partial_t \psi = \boldsymbol{\alpha} \cdot (-i\boldsymbol{\nabla} - e\mathbf{A})\psi + m\beta\psi + eA^0\psi,$$

$$(c^{-2}\partial_t^2 - \Delta)A^0 = e\psi^*\psi, \qquad (c^{-2}\partial_t^2 - \Delta)\mathbf{A} = e\psi^*\boldsymbol{\alpha}\psi.$$

Here $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$, and α^j and β are the 4×4 Dirac matrices:

$$\alpha^j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

with $\{\sigma_j\}_{j=1}^3$ the Pauli matrices. We will not distinguish lower and upper indices j of α and σ, so that $\alpha_j = \alpha^j$, $\sigma_j = \sigma^j$. The α-matrices and γ-matrices are related by

$$\gamma^j = \beta\alpha^j, \quad 1 \le j \le 3; \qquad \gamma^0 = \beta.$$

The solitary wave $(\phi e^{-i\omega t}, A^\mu(x))$ satisfies the stationary system

$$\omega\phi = \boldsymbol{\alpha}\cdot(-i\boldsymbol{\nabla}-e\mathbf{A})\phi+m\beta\phi+eA^0\phi, \qquad -\boldsymbol{\Delta}A^\mu = e\bar{\phi}\gamma^\mu\phi.$$

Theorem 3. *There exists* $\omega_* > -m$ *such that for* $\omega \in (-m, \omega_*)$ *there is a solution to this system of the form*

$$\phi(x,\omega) = \begin{bmatrix}\epsilon^3\Phi_1(\epsilon x,\epsilon)\\ \epsilon^2\Phi_2(\epsilon x,\epsilon)\end{bmatrix}, \qquad \epsilon = \sqrt{m^2-\omega^2},$$

with

$$\Phi = \begin{bmatrix}\Phi_1\\ \Phi_2\end{bmatrix} \in C^\infty\Big((0,\epsilon_*)\,;\ \Big(H^2(\mathbf{R}^3;\mathbb{C}^2)\oplus H^2(\mathbf{R}^3;\mathbb{C}^2)\Big)\Big), \qquad \epsilon_*$$

and with

$$A^\mu \in C^\infty\Big((0,\epsilon_*)\,;\ \dot{H}^1(\mathbf{R}^3,\mathbf{R})\cap L^\infty(\mathbf{R}^3,\mathbf{R})\Big), \qquad 0\le\mu\le 3.$$

Above, $\dot{H}^1 = \dot{H}^1(\mathbf{R}^3,\mathbf{R})$ *is the homogeneous Dirichlet space of* L^6 *functions with* $\|f\|^2_{\dot{H}^1} = \int |\nabla f|^2\,dx < \infty$*. For small* $\epsilon > 0$*, one has*

$$\|\Phi_1 - \hat{\Phi}_1\|_{H^2} + \|\Phi_2 - \hat{\Phi}_2\|_{H^2} = O(\epsilon^2),$$

where $\hat{\Phi}_1(y)$*,* $\hat{\Phi}_2(y)$ *are of Schwartz class. The solutions can be chosen so that in the nonrelativistic limit* $\epsilon = 0$ *one has*

$$\hat{\Phi}_2(y) = \varphi_0(y)\boldsymbol{n}, \qquad \hat{\Phi}_1(y) = \frac{i}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\hat{\Phi}_2(y), \quad (6)$$

where $\boldsymbol{n} \in \mathbb{C}^2$, $|\boldsymbol{n}| = 1$, and φ_0 is a strictly positive spherically symmetric solution of Schwartz class to the Choquard equation

$$-\frac{1}{2m}\varphi_0 = -\frac{1}{2m}\Delta\varphi_0 - \Big(\frac{2}{4\pi|x|} * \varphi_0^2\Big)\varphi_0, \qquad \varphi_0(x) \in \mathbf{R}, \quad x \in \tag{7}$$

The Dirac field ϕ has exponential decay, while the electromagnetic potential satisfies

$$A^0(x) = \frac{\|\phi\|_{L^2}^2}{4\pi|x|} + O(\langle x\rangle^{-2}), \qquad \mathbf{A}(x) = O(\langle x\rangle^{-2})$$

as $|x| \to +\infty$.

- Solutions axi- (not radially) symmetric
- Paradoxical repulsion/attraction behaviour indicates single particle Dirac equation contains positronic element. Semi-classical limit has such “mixed” characteristics.
- Need analysis of bound states in quantum field theory to assess likely physical significance (if any).

Motivation to study the Schwinger model

- Understand significance of bound states in quantum field theory?
- Quantum electrodynamics in $1+3$ dimensions not known to exist; may not exist in usual mathematical sense. (QCD)
- Interesting mathematical structure and physical features: mass generation, gauge invariance issues, fermion/boson transformation...
- But: has no nonrelativistic limit - need massive fermions for that.

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Gauge Invariance and Mass. II*

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The possibility that a vector gauge field can imply a nonzero mass particle is illustrated by the exact solution of a one-dimensional model.

It has been remarked[1] that the gauge invariance of a vector field does not necessarily require the existence of a massless physical particle. In this note we shall add a few related comments and give a specific model for which an exact solution affirms this logical possibility. The model is the physical, if unworldly situation of electrodynamics in one spatial dimension, where the charge-bearing Dirac field has no associated mass constant. This example is rather unique since it is a simple model for which there is an exact divergence-free solution.[2]

GENERAL DISCUSSION

The Green's function of an Abelian vector gauge field has the structure

$$\mathcal{G}_{\mu\nu}(x,x')=\pi_{\mu\nu}(-i\partial)\mathcal{G}(-i\partial)\delta(x-x'),$$

where $\pi_{\mu\nu}(p)$ is a gauge-dependent projection matrix and

$$\mathcal{G}(p)=\int_0^\infty dm^2\,\frac{B(m^2)}{p^2+m^2-i\epsilon},$$

which is subject to the sum rule

$$1=\int_0^\infty dm^2\,B(m^2).$$

An alternative form of $\mathcal{G}(p)$ is

$$\mathcal{G}(p)=\left[p^2+\lambda^2-i\epsilon+(p^2-i\epsilon)\int_0^\infty dm^2\,\frac{s(m^2)}{p^2+m^2-i\epsilon}\right]^{-1},$$

where the function $s(m^2)$ and the constant λ^2 are nonnegative. The latter has been derived[3] with the understanding that the pole at $z=0$ of the expression

$$-\frac{\lambda^2}{z}+\int_0^\infty dm^2\,\frac{s(m^2)}{m^2-z}=\int_0^\infty \frac{dm^2}{m^2-z}[s(m^2)+\lambda^2\delta(m^2)]$$

is completely described by the parameter λ. Accordingly,

$$\mathcal{G}(0)=\frac{1}{\lambda^2}=\int_0^\infty dm^2\,\frac{B(m^2)}{m^2},$$

and $\lambda^2>0$ unless $m=0$ is contained in the spectrum. Thus, it is necessary that λ vanish if $m=0$ is to appear as an isolated mass value in the physical spectum. But it is also necessary that

$$s(0)=0,$$

such that

$$\int_{\to 0}^\infty \frac{dm^2}{m^2}s(m^2)<\infty,$$

for only then do we have a pole at $p^2=0$,

$$p^2\sim 0:\quad \mathcal{G}(p)\sim B_0/(p^2-i\epsilon),\quad 0<B_0<1.$$

Under these conditions,

$$B(m^2)=B_0\delta(m^2)+B_1(m^2),$$

where

$$B_0=\left(1+\int_0^\infty \frac{dm^2}{m^2}s(m^2)\right)^{-1}$$

and

$$B_1(m^2)=[s(m^2)/m^2]\Big/\left\{\left[1+P\int_0^\infty dm'^2\frac{s(m'^2)}{m'^2-m^2}\right]^2+[\pi s(m^2)]^2\right\}.$$

The physical interpretation of $s(m^2)$ derives from the relation of the Green's function to the vacuum transformation function in the presence of sources. For sufficiently weak external currents $J_\mu(x)$,

$$\langle 0|0\rangle^J=\exp\left[\tfrac{1}{2}i\int(dx)(dx')J^\mu(x)\mathcal{G}_{\mu\nu}(x,x')J^\nu(x')\right]$$

$$=\exp\left[\tfrac{1}{2}i\int(dp)J^\mu(p)^*\mathcal{G}(p)J_\mu(p)\right],$$

which involves the reduction of the projection matrix $\pi_{\mu\nu}(p)$ to $g_{\mu\nu}$ for a conserved current, or equivalently

$$p_\mu J^\mu(p)=0.$$

We shall present this transformation function as a measure of the response to the external vector potential

$$A_\mu(p)=\mathcal{G}(p)J_\mu(p),$$

namely,

$$\langle 0|0\rangle^J=\exp\left[\tfrac{1}{2}i\int(dp)A^\mu(p)^*\mathcal{G}(p)^{*-1}A_\mu(p)\right].$$

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[1] J. Schwinger, Phys. Rev. **125**, 397 (1962).

[2] There is a divergence in the so-called Thirring model [W. E. Thirring, Ann. Phys. (New York) **3**, 91 (1958)], which uses local current interactions rather than a Bose field.

[3] J. Schwinger, Ann. Phys. (New York) **9**, 169 (1960).

The probability that the vacuum state shall persist despite the disturbance is

$$|\langle 0|0\rangle^J|^2=\exp\left[-\int (dp)A_\mu(p)^*A_\mu(p)\ \mathrm{Im}\mathcal{G}(p)^{*-1}\right]$$
$$=\exp\left[-\pi\int (dp)dm^2\delta(p^2+m^2)\times s(m^2)(-\tfrac{1}{2})F^{\mu\nu}(p)^*F_{\mu\nu}(p)\right],$$

which exhibits $s(m^2)$ as a measure of the probability that an external field $F_{\mu\nu}$ will produce a vacuum excitation involving an energy-momentum transfer measured by the mass m.

The vanishing of $s(m^2)$ at $m=0$ is normal threshold behavior for an excitation function. If a zero-mass particle is not to exist, $m=0$ must be an abnormal threshold. Two possibilities can be distinguished. In the first of these, $s(m^2)$ is finite or possibly singular at $m=0$, but in such a way that

$$\lim_{z\to 0} z\int_0^\infty dm^2\frac{s(m^2)}{m^2-z}=0.$$

Then the physical mass spectrum begins at $m=0$ but there is no recognizable zero-mass particle. For the second situation, $s(m^2)$ has a delta-function singularity at $m^2=0$,

$$s(m^2)=\lambda^2\delta(m^2)+s_1(m^2),$$

and

$$s_1(m^2)=0,\quad m^2<m_0^2.$$

If the threshold mass m_0 is zero, the restriction of the previous situation applies to the function $s_1(m^2)$. Now, $m=0$ is not contained in the spectrum at all. This statement is true even if $m_0=0$ for, according to the structure of $B_1(m^2)=B(m^2)$,

$$B(m^2)=\frac{m^2s_1(m^2)}{[R(m^2)]^2+[\pi m^2 s_1(m^2)]^2},$$

in which

$$R(m^2)=m^2-\lambda^2+m^2P\int_{m_0^2}^\infty dm'^2\frac{s_1(m'^2)}{m'^2-m^2},$$

we have

$$\lim_{m^2\to 0}B(m^2)=\lim_{m^2\to 0}\frac{m^2s_1(m^2)}{\lambda^4}=0.$$

Let us suppose that m_0 is the threshold of a continuous spectrum. A stable particle of mass $m<m_0$ will exist if $R(m_0^2)>0$. Should both $R(m_0^2)$ and $s_1(m_0^2)$ be zero there would be a stable particle of mass m_0. No stable particle exists if $R(m_0^2)<0$. But there is always an unstable particle, in a certain sense. By this we mean that $R(m^2)$ vanishes at some mass value $m_1>m_0$, under the general restrictions required for the continuity of the function $R(m^2)$, as a consequence of this function's asymptotic approach to $+\infty$ with increasing m^2. The mass m_1 will be physically recognizable as the mass of an unstable particle if the mass width

$$\gamma=\frac{\pi m_1 s_1(m_1^2)}{[dR(m_1^2)/dm_1^2]}$$

is sufficiently small. [We take the derivative of $R(m_1^2)$ to be positive, which is appropriate for the simplifying assumption that only one zero occurs.] The contribution of such a fairly sharp resonance to the sum rule for $B(m^2)$ is given by

$$\int_{m\sim m_1}dm^2B(m^2)=[dR(m_1^2)/dm_1^2]^{-1}<1.$$

SIMPLE MODELS

Some of these possibilities can be illustrated in very simple physical contexts. We consider the linear approximation to the problem of electromagnetic vacuum polarization for spaces of dimensionality $n=2$ and 1. A modification of a technique[4] previously applied to three-dimensional space yields for $m>m_0$:

$$s(m^2)=\int_0^{(1-m_0^2/m^2)^{1/2}}dv(1-v^2)(e^2/8\pi^2)\quad\text{for}\quad n=3$$
$$=\int_0^{(1-m_0^2/m^2)^{1/2}}dv(1-v^2)(e^2/4\pi^2)\times[m^2(1-v^2)-m_0^2]^{-1/2}\quad\text{for}\quad n=2$$
$$=\int_0^{(1-m_0^2/m^2)^{1/2}}dv(1-v^2)(e^2/\pi)\delta[m^2(1-v^2)-m_0^2]\quad\text{for}\quad n=1;$$

for $m<m_0$:

$$s(m^2)=0,$$

where the known result for $n=3$ has been included for comparison. The threshold mass m_0 is that for single pair creation. It should be noted that the coupling constant e^2 of electrodynamics in n-dimensional space has the dimensions of a mass raised to the power $3-n$. For $n<3$ this single pair approximation does not lead to difficulties concerning the existence of such integrals as

$$B_0^{-1}-1=\int_0^{\to\infty}\frac{dm^2}{m^2}B(m^2),$$

since, for $m\gg m_0$:

$$\begin{aligned}s(m^2)&\sim(e^2/12\pi^2) &&\text{for}\quad n=3,\\ &\sim(e^2/16\pi)(1/m) &&\text{for}\quad n=2,\\ &\sim(e^2/2\pi)(m_0^2/m^4) &&\text{for}\quad n=1.\end{aligned}$$

The particular situation in which we are interested appears at the limit $m_0\to 0$. Then we have

$$\begin{aligned}s(m^2)&=(e^2/16\pi)(1/m) &&\text{for}\quad n=2,\\ &=(e^2/\pi)\delta(m^2) &&\text{for}\quad n=1.\end{aligned}$$

[4] *Selected Papers on Quantum Electrodynamics* (Dover Publications, New York, 1958), p. 209.

Two-dimensional electrodynamics illustrates the first of the two possibilities for an anomalous threshold at $m=0$. The spectral function $B(m^2)$ describes a purely continuous spectrum,

$$dm^2\, B(m^2)=\frac{2}{\pi}\frac{e^2}{16}\frac{dm}{m^2+(e^2/16)^2},$$

and an m integration from 0 to ∞ satisfies the sum rule. In one-dimensional electrodynamics we meet a special case of the second possibility, with

$$\lambda^2=e^2/\pi, \quad s_1(m^2)=0.$$

Accordingly,

$$B(m^2)=\delta(m^2-(e^2/\pi))$$

and the mass spectrum is localized at one point, describing a stable particle of mass $e/\pi^{1/2}$.

The basis indicated for the latter conclusion will not be very convincing, but it is an exact result. To prove this we first compute for one spatial dimension the electric current induced by an arbitrary external potential in the vacuum state of a massless charged Dirac field. The appropriate gauge-invariant expression for the current[5] is

$$j_\mu(x)=-\tfrac{1}{2}e\,\mathrm{tr}q\alpha_\mu G(x,x')\exp\left[-ieq\int_{x'}^{x}d\xi^\mu\,A_\mu(\xi)\right]\Bigg|_{x'\to x},$$

in which the approach of x' to x is performed from a spatial direction in order to maintain time locality. The Green's function is defined by the differential equation

$$\alpha^\mu[\partial_\mu-ieqA_\mu(x)]G(x,x')=\delta(x-x'),$$

together with the outgoing wave boundary condition, in the absence of the potential. Only two Dirac matrices appear here, $\alpha^0=-\alpha_0=1$ and $\alpha^1=\alpha_1$, which has the eigenvalues ±1. Those are also the eigenvalues of the independent charge matrix q. The Green's function equation can be satisfied by writing

$$G(x,x')=G^0(x,x')\exp\{ieq[\phi(x)-\phi(x')]\},$$

where

$$\alpha^\mu\partial_\mu\phi=\alpha^\mu A_\mu(x)$$

and

$$\alpha^\mu\partial_\mu G^0(x,x')=\delta(x-x').$$

The latter defines the free Green's function, which is given explicitly by

$$G^0(x,x')=\int_0^\infty\frac{dp}{2\pi}\exp[ip\alpha^\mu(x_\mu-x_\mu')] \quad \text{for} \quad x^0>x^{0\prime},$$

$$=-\int_{-\infty}^0\frac{dp}{2\pi}\exp[ip\alpha^\mu(x_\mu-x_\mu')] \quad \text{for} \quad x^0<x^{0\prime}.$$

[5] The necessity for the line integral factor has been noted before [J. Schwinger, Phys. Rev. Letters **3**, 296 (1959)].

At equal times, and for sufficiently small x_1-x_1', we have

$$G(x,x')\exp\left[-ieq\int_{x'}^{x}d\xi^\mu\,A_\mu(\xi)\right]$$

$$\cong\frac{i}{2\pi}\frac{\alpha_1}{x_1-x_1'}-\frac{eq}{2\pi}\alpha_1[\partial_1\phi(x)-A_1(x)].$$

The first term does not contribute to the vacuum current when the limit $x_1'\to x_1$ is performed symmetrically. On utilizing the relation

$$\alpha_1(\partial_1\phi-A_1)=-(\partial_0\phi-A_0),$$

we find that

$$j_\mu(x)=-\frac{e^2}{\pi}A_\mu(x)+\partial_\mu\left[\frac{e^2}{4\pi}\,\mathrm{tr}\phi(x)\right].$$

This expression for the induced current is Lorentz covariant, gauge invariant, and obeys the equation of conservation. It is also a linear function of the external field. To verify these statements we construct a differential equation for $\mathrm{tr}\phi(x)$ by multiplying the ϕ equation with $\partial_0-\alpha_1\partial_1$ and evaluating the trace. The result is

$$\partial^2\tfrac{1}{4}\,\mathrm{tr}\phi(x)=\partial_\mu A^\mu(x),$$

and therefore

$$\tfrac{1}{4}\,\mathrm{tr}\phi(x)=-\int(dx')D(x,x')\partial_\mu' A^\mu(x'),$$

in which D is the outgoing-wave Green's function defined by

$$-\partial^2 D(x,x')=\delta(x-x').$$

By using a symbolic matrix notation for coordinates and vector indices, we can write

$$j=-(e^2/\pi)(1+\partial D\partial)A,$$

which exhibits the symmetrical projection matrix

$$\pi=1+\partial D\partial,$$
$$\partial\pi=\pi\partial=0,$$

that guarantees gauge invariance and current conservation.

We shall insert this result in the functional differential equation obeyed by the Green's functional $G[J]$, the vacuum transformation function in the presence of external currents. It is convenient to use the particular system of equations that refer to the Lorentz gauge,

$$\left\{(\partial\partial-\partial^2)\frac{1}{i}\frac{\delta}{\delta J}-(1+\partial D\partial)\left[J+j\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right]\right\}G[J]=0,$$

$$\partial\frac{\delta}{\delta J}G[J]=0,$$

which also utilize a symbolic notation for vectorial coordinate functions. We have written $j(-i\delta/\delta J)$ to indicate the conversion of $j(A)$ into a functional differential operator by the substitution $A\rightarrow -i\delta/\delta J$. The functional differential equation implied by the known structure of this operator is

$$\pi\left[\left(-\partial^2+\frac{e^2}{\pi}\right)\frac{1}{i}\frac{\delta}{\delta J}-J\right]G[J]=0,$$

or, on uniting the two defining properties of the functional,

$$\left(\frac{1}{i}\frac{\delta}{\delta J}-\pi\mathcal{G}J\right)G[J]=0,$$

in which

$$[-\partial^2+(e^2/\pi)]\mathcal{G}(x,x')=\delta(x-x').$$

The Green's functional $G[J]$ is therefore given exactly by

$$G[J]=\exp\left[\tfrac{1}{2}i\int(dx)(dx')J^\mu(x)\mathcal{G}_{\mu\nu}(x,x')J^\nu(x')\right],$$

with

$$\mathcal{G}_{\mu\nu}(x,x')=\pi_{\mu\nu}(-i\partial)\mathcal{G}(-i\partial)\delta(x-x')$$

and

$$\mathcal{G}(p)=\frac{1}{p^2+(e^2/\pi)-i\epsilon}.$$

Thus, all states that can be excited by vector currents are fully described as noninteracting ensembles of Bose particles with the mass $e/\pi^{1/2}$.

Concerning the complete Green's functional including Fermi sources, $G[\eta J]$, we shall only remark that

$$G[\eta J]=\exp\left[-\frac{1}{2}\int(dx)(dx')\eta(x)\times G\left(x,x',\frac{1}{i}\frac{\delta}{\delta J}\right)\eta(x')\right]G[J],$$

in which the Green's function can be presented as

$$G(x,x',A)=G^0(x,x')\exp\left[i\int(d\xi)j^\mu(\xi,x,x')A_\mu(\xi)\right]$$

with

$$j^\mu(\xi,x,x')=eq\alpha^\mu\left(\alpha^1\frac{\partial}{\partial\xi^1}-\frac{\partial}{\partial\xi^0}\right)[D(\xi,x)-D(\xi,x')].$$

On expanding the Green's functional in even powers of the Fermi source, we encounter functional differential operators that are contained in one or more factors of the type

$$\exp\left[\int(d\xi)j^\mu(\xi,x,x')\delta/\delta J^\mu(\xi)\right],$$

the effect of which is simply to produce the translation $J\rightarrow J+j$ in $G[J]$. The first Fermi Green's function is

$$G(x,x')=G(x,x',-i\delta/\delta J)G[J]|_{J=0}$$
$$=G^0(x,x')\exp\left[\tfrac{1}{2}i\int(d\xi)(d\xi')\times j^\mu(\xi,x,x')\mathcal{G}_{\mu\nu}(\xi,\xi')j^\nu(\xi',x,x')\right].$$

The latter exponential factor is given by

$$\exp\left[-\frac{i}{4\pi}\int(dp)\left(\frac{1}{p^2-i\epsilon}-\frac{1}{p^2+(e^2/\pi)-i\epsilon}\right)\times(1-e^{ip(x-x')})\right].$$

We shall be content to note that this integral and the similar integrals encountered in more general Green's functions are completely convergent. The detailed physical interpretation of the Green's functions is rather special and apart from our main purpose.

These simple examples are quite uninformative in one important respect. They do not exhibit a critical dependence upon the coupling constant. As we have discussed previously, one can view the electromagnetic field as undercoupled and the hypothetical vector field that relates to nucleonic charge as overcoupled, in the sense of a critical value at which the massless Bose particle ceases to exist. The corresponding appearance of an anomalous zero-mass threshold must be attributed to a dynamical mechanism. We can supply an artificial mathematical model that illustrates the situation. Let the following be a contributory term in $s(m^2)$:

$$s_0(m^2)=\frac{\lambda^2}{\pi}\frac{m\gamma}{(m^2-m_0^2\kappa)^2+(m\gamma)^2},$$

in which m_0 is a characteristic physical fermion mass, and λ/m_0, γ/m_0, and κ are positive functions of the (dimensionless) coupling constant. In electrodynamics the near-resonant contributions of such a term can be identified with the creation of a unit angular momentum positronium state, while the values far below resonance refer to the creation of three-photon states (the model falsifies the latter, which should vary as m^8 for $m\ll m_0$). It is reasonable to suppose that κ decreases with increasing strength of the coupling, and we can imagine that a critical value exists for which both κ and γ reach zero, with finite λ. In that circumstance,

$$s_0(m^2)=\lambda^2\delta(m^2),$$

and the null-mass particle disappears from the spectrum. Since this argument requires that one type of excitation move down to zero mass at the critical coupling strength, it is plausible that some other types of excitation will then be located at fairly small fractions of m_0. Thus, one could anticipate that the known spin-0 bosons, for example, are secondary dynamical manifestations of strongly coupled primary fermion fields and vector gauge fields. This line of thought emphasizes that the question "Which particles are fundamental?" is incorrectly formulated. One should ask "What are the fundamental fields?"

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Scattering of Electromagnetic Waves in Saxon-Schiff Theory

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We calculate the diffraction of electromagnetic waves by weak scatterers with complex dielectric constant and permeability using the Saxon-Schiff theory of potential scattering. Boundary conditions, polarizations, and the optical theorem are discussed to some extent. Our results for the scattering amplitude contain certain special cases obtained previously by other authors. In an Appendix, we compare the results for the scattering by a homogeneous dielectric sphere with those of the exact Mie theory. It is seen that the Saxon-Schiff theory gives a good qualitative agreement insofar as it reproduces the diffraction maxima and minima, in vast superiority to the Born approximation. In the asymptotic limit $kR\to\infty$, the radar cross section is shown to agree with the exact result for a not too large index of refraction.

THE theory of Saxon and Schiff,[1] originally developed for high-energy scalar potential scattering, has been applied to the scattering of electromagnetic waves by dielectric bodies.[2] Schiff[3] has also considered scattering of vector waves using an earlier version of the theory, valid for either small or large angles only. In this note, we derive the scattering amplitude of electromagnetic waves for a general weak scatterer with complex dielectric constant and permeability, and demonstrate that the results can be made to reduce to the large- and small-angle expressions of Schiff[3] in the respective limits.

Maxwell's equations, setting $c=1$ and assuming a harmonic time dependence of the fields,

$$\sim\exp(-ikt),$$

become

$$\nabla\times\mathbf{E}=ik\mu\mathbf{H},\quad \nabla\times\mathbf{H}=(\sigma-ik\epsilon)\mathbf{E}. \tag{1}$$

No free charges are assumed to be present; σ is the conductivity, and ϵ, μ are dielectric constant and permeability, respectively (we shall use Gaussian units, $\epsilon_0=\mu_0=1$). Taking the divergence of the second equation, we get

$$\nabla\cdot\epsilon'\mathbf{E}=0, \tag{2}$$

where we have introduced the complex dielectric constant,

$$\epsilon'=\epsilon(1+i\nu),$$

with

$$\nu=\sigma/k\epsilon.$$

Elimination of $\mathbf{H}$ from (1) gives the wave equation

$$\nabla^2\mathbf{E}+K^2\mathbf{E}=\nabla\nabla\cdot\mathbf{E}-\mu^{-1}\nabla\mu\times(\nabla\times\mathbf{E}), \tag{3}$$

with the squared propagation constant

$$K^2=k^2\mu\epsilon'. \tag{4}$$

Equation (2) can again be obtained by taking the divergence of the wave equation.

Following reference (1), a Green's function

$$F(\mathbf{r},\mathbf{r}')=F(\mathbf{r}',\mathbf{r})=-(4\pi\rho)^{-1}e^{iS(\mathbf{r},\mathbf{r}')} \tag{5}$$

will be considered, where

$$\rho=|\mathbf{r}-\mathbf{r}'|;$$

the phase is assumed to have the limits

$$\lim_{\mathbf{r}'\to\mathbf{r}}\rho^{-1}S(\mathbf{r},\mathbf{r}')=C(\mathbf{r}),\qquad \lim_{r\to\infty}\nabla S=k\mathbf{n}+O(r^{-1});\quad \mathbf{r}=\mathbf{n}r. \tag{6}$$

This Green's function satisfies the differential equation

$$\nabla^2F+(\nabla S)^2F=\delta(\mathbf{r}-\mathbf{r}')+iF\rho^2\nabla\cdot(\rho^{-2}\nabla S). \tag{7}$$

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The Schwinger Model Quantum field theory in 1+1 dimensions, zero mass for Dirac fermion:

$$S = \int -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} i \not{D}_a \psi \;\; dx\, dt\,, \qquad \not{D}_a = \gamma^\mu(\partial_\mu - iea_\mu)$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

Phase invariance: $\psi \to e^{ig}\psi$

Chiral invariance: $\psi \mapsto e^{i\gamma^5\theta}\psi$.

Associated conservation laws:

$$j = j^\mu = \overline{\psi}\gamma^\mu\psi = (j^0, j^1)$$

$$j^{5,\mu} = \overline{\psi}\gamma^\mu\gamma^5\psi = (j^1, j^0)$$

are classically conserved ($\partial_\mu j^\mu = 0 = \partial_\mu j^{5,\mu}$), but:

> *In the quantum theory only the first of these properties holds - there is an anomaly in the chiral conservation law.*

The precise mathematical expression of the anomaly is

$$\partial_\mu j^{5,\mu} = -\frac{1}{\pi} E,$$

where $E = \dot{a} - \partial a_0$ is the electric field. This is an equality of operator valued distributions.

Mass generation and Bosonization Together with equation of motion (Maxwell) this implies

$$(\Box + \frac{e^2}{\pi})E = 0$$

in place of $\Box E = 0$ - mass generation. Crucial issues which arise:

- Requirement of energy bounded below $\Longrightarrow$ introduce fermionic Fock space with (non-interacting) vacuum Ω_0 on which gauge group acts non trivially
- Enforcing gauge invariance leads to modified Hamiltonian and breaking of chiral symmetry
- Existence of gauge invariant (interacting vacuum) Ψ_0 but at expense of chiral anomaly.
- Anomaly+bosonization leads to real scalar field with mass $e/\sqrt{\pi}$:

$$H_S = \frac{1}{2}\int_0^L \left(\Pi(x)^2 + \partial\Phi(x)^2 + \frac{e^2}{\pi}\Phi(x)^2\right) dx\,,$$

  to be expected from Schwinger's work (1962).
- Fundamental excitations are bound states of fermion/anti-fermion

Gauge invariance and large gauge transformations Phase invariance localises to gauge invariance: $\psi \to e^{ig}\psi$ and $a_\mu \to a_\mu + \partial_\mu g$ where $g = g(t,x)$ is a sufficiently regular function *which is L periodic in x*. Coulomb gauge: choose periodic g so that spatial component $a_1 = a$ is constant: cannot be made zero with periodic g. This leave residual gauge invariance by the group

$$\mathbb{Z} = \{g_N(x) = e^{2\pi i N x/L}\}_{N\in\mathbb{Z}}$$

of *large* or *modular* gauge transformations.

- a defined mod $2\pi/L$ - takes values in the circle $S^1 = \mathbf{R}/(2\pi/L)$ which is dual to the spatial domain $\mathbf{R}/L$.
- careful treatment of the action of the group $\mathbb{Z}$ illuminates greatly the role of gauge invariance in producing the anomaly and the interacting vacuum.
- Noninteracting Fock vacuum is not gauge invariant ; physical (interacting) vacuum is gauge invariant.
- Vacuum of YM_4?

Hamiltonian and Quantization I

Fermi procedure leads to Hamiltonian

$$\int_0^L \frac{1}{2e^2}\dot{a}^2 - \psi^\dagger\Big(i\gamma^5(\partial - ia)\psi\Big) + \frac{1}{2}e^2(\psi^\dagger\psi)(-\Delta)^{-1} * (\psi^\dagger\psi)\ \ dx\,.$$

Here $(-\Delta)^{-1}$ means the kernel of the operator $-\Delta = -\partial^2$ on $[0, L]$ with periodic boundary conditions, $*$ is convolution and $\partial = \partial_x$. Longitudinal component of the electric field has been integrated out leaving only the transverse component $E^{tr} = \dot{a}$.

Associated classical equations of motion:

$$\begin{aligned} i\dot{\psi} &= -i\gamma^5(\partial\psi - ia\psi) - a_0\psi \\ \dot{E}^{tr} &= \frac{e^2}{L}\int_0^L \psi^\dagger\gamma^5\psi\, dx\,, \qquad \dot{a} = E^{tr}\ , \end{aligned} \tag{8}$$

where a_0 is determined by the Gauss law constraint $-\Delta a_0 = -e^2\,\psi^\dagger\psi = -e^2\,j^0$. We will write $j^0 = \psi^\dagger\psi$ and $j^1 = \psi^\dagger\gamma^5\psi$ for the currents and $Q = \int_0^L j^0\,dx\,, Q^5 = \int_0^L j^1\,dx$ for the corresponding charges.

To quantize the theory it is necessary to associate operators to the fields which satisfy the canonical relations:

$$\{\psi_\alpha(t,x), \psi_\beta^\dagger(t,y)\} = \delta_{\alpha\beta}\delta(x-y) \tag{9}$$

(other anti-commutators being zero), and

$$[E^{tr}, a] = [\dot{a}, a] = -\frac{ie^2}{L} \tag{10}$$

(other commutators being zero).

Energy Levels

In Coulomb gauge with $a = constant$ the operator

$$-i\gamma^5(\partial\psi - ia\psi)$$

can be diagonalized :

$$\gamma^5 u^R = u^R \quad \text{and} \quad \gamma^5 u^L = -u^L \, .$$

giving spectrum

$$\pm\,(k_n + a)\,, \quad k_n = \frac{2n\pi}{L} \qquad n \in \mathbb{Z}$$

Second quantization with Dirac sea leads to Hamiltonian

$$\sum (k_n + a)\, a^\dagger_{n,R} a_{n,R} \, - \, \sum (k_n + a)\, a^\dagger_{n,L} a_{n,L}$$

and charge operator

$$\sum \, a^\dagger_{n,L} a_{n,L} \, + \, \sum \, a^\dagger_{n,R} a_{n,R}$$

commutation relations

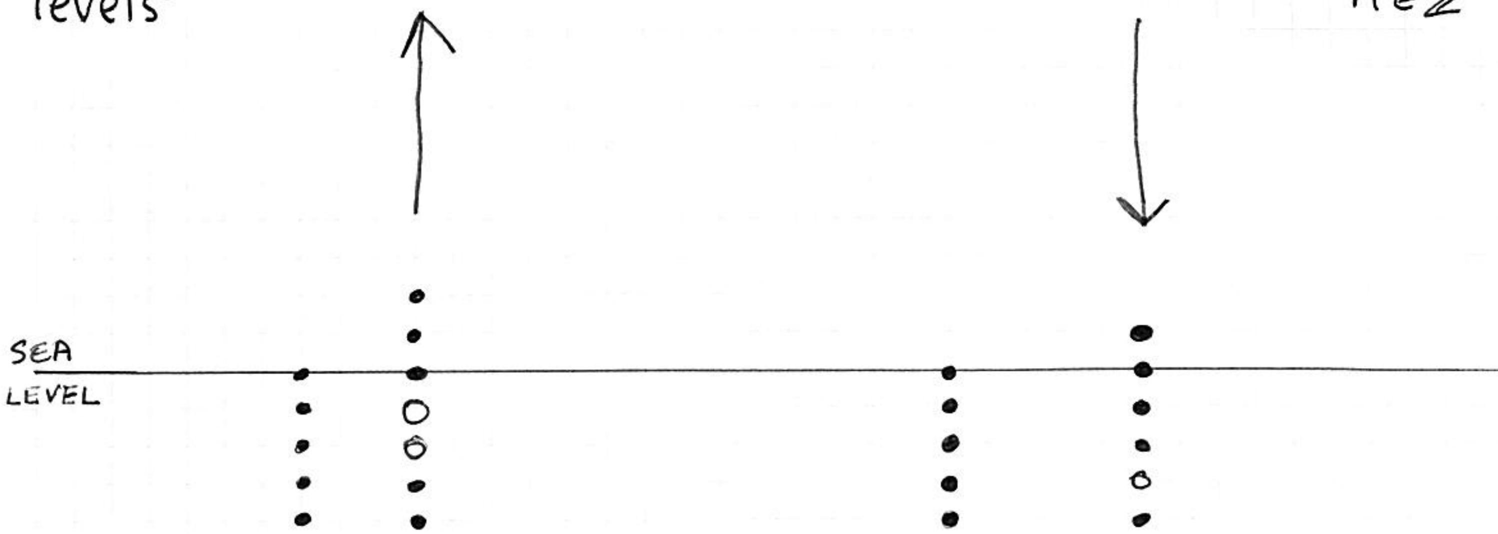
$$\{a_{n,L}, a^\dagger_{m,L}\} = \delta_{nm} \qquad \{a_{n,R}, a^\dagger_{m,R}\} = \delta_{nm}$$

DIRAC SEA + ACTION OF $\mathbb{Z}$

	LEFT	RIGHT
Energy levels	$k_n + a$	$-(k_n + a)$

$k_n = \frac{2n\pi}{L}$

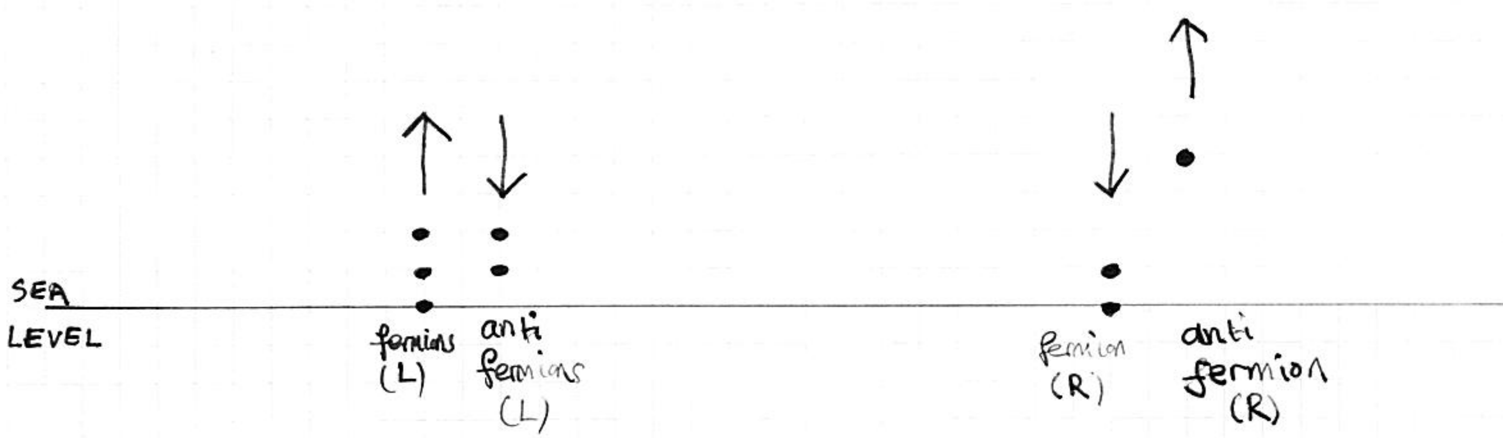
$n \in \mathbb{Z}$



o = unoccupied state (HOLE)

IN POSITIVE ENERGY REPRESENTATION:

HOLES TURN INTO ANTI-FERMIONS



ARROWS GIVE ACTION OF GAUGE TRANSFORMATIONS $\mathbb{Z}$

EITHER CREATE LEFT ELECTRON
+ DESTROYS RIGHT ELECTON

OR CREATE ELECRON (L) + POSITRON (R) PAI

In <u>positive energy representation</u> define

$$b_n = a_n^R \quad (n \geq 0) \qquad b_n = a_n^L \quad (n < 0)$$

and

$$c_n = a_{-n}^{R,\dagger} \quad (n > 0) \qquad c_n = a_{-n}^{L,\dagger} \quad (n \leq 0)$$

then

$$\sum_{m \in \mathbb{Z}} |k_m| \, (b_m^\dagger b_m + c_m^\dagger c_m) \; - \; a Q^5$$

$$Q^5 = \sum_{n \geq 0} b_n^\dagger b_n - \sum_{n<0} b_n^\dagger b_n - \sum_{n>0} c_n^\dagger c_n + \sum_{n \leq 0} c_n^\dagger c_n$$

while charge operator is

$$Q \; = \; \left(b_n^\dagger b_n - c_n^\dagger c_n \right) .$$

- Energy now bounded below, BUT
- At expense of introducing vacuum which is not gauge invariant

Quantization of electric field Use the Schrödinger representation in which $a_1 = a$ is represented by coordinate multiplication on $L^2([0, \frac{2\pi}{L}])$, while

$$E^{tr} = -\frac{ie^2}{L}\frac{d}{da}.$$

In the absence of interaction with any matter fields the electromagnetic field is described by the Hamiltonian

$$H_{em} = -\frac{e^2}{2L}\frac{d^2}{da^2}$$

on $L^2([0, \frac{2\pi}{L}])$ with *periodic* boundary conditions

- The large gauge transformations $\mathbb{Z}$ are the reason that periodic boundary conditions are appropriate
- Periodic boundary conditions have to be modified in the presence matter.

Positive energy representation Interpret the relations (9) :

$$\psi = \frac{1}{\sqrt{L}} \sum_{n\in\mathbb{Z}} \Big(b_n u_n e^{ik_n x} + c_n^\dagger v_n e^{-ik_n x} \Big) , \quad k_n = \frac{2n\pi}{L} \quad (11)$$

with

$$\{b_n, b_{n'}^\dagger\} = \{c_n, c_{n'}^\dagger\} = \delta_{nn'} \quad (12)$$

(other anti-commutators being zero) and

$$\begin{aligned} u_n &= u^R \mathbb{1}_{\{n\geq 0\}} + u^L \mathbb{1}_{\{n<0\}} , \\ v_n &= u^R \mathbb{1}_{\{n>0\}} + u^L \mathbb{1}_{\{n\leq 0\}} . \end{aligned} \quad (13)$$

The $u^{L,R}$ are eigenvectors of γ^5 with $\gamma^5 u^R = u^R$ and $\gamma^5 u^L = -u^L$. The $b_m^\dagger$, b_m (resp. $c_m^\dagger$, c_m) are fermionic (resp. anti-fermionic) creation, annihlation operators acting on the zero charge fermionic Fock space $\mathcal{H}_0$.

Recall the fermionic Fock space: there is a (non-interacting) vacuum Ω_0 and associated finite particle states

$$\Omega_{\mathbf{m},\mathbf{n}} = \prod b_{m_i}^\dagger c_{n_j}^\dagger \Omega_0 \quad (14)$$

where $\mathbf{m} = \{m_i\}_{i=1}^M$ and $\mathbf{n} = \{n_j\}_{j=1}^N$ range over subsets of $\mathbb{Z}$ of arbitrary finite size. Let $\mathcal{F}$ be the linear span of all the $\Omega_{\mathbf{m},\mathbf{n}}$, let $\mathcal{F}_0 \subset \mathcal{F}$ be the zero charge subspace in which there are equal numbers of fermions and anti-fermions, i.e. $M = N$. The zero charge Fock space $\mathcal{H}_0$ is the completion of $\mathcal{F}_0$ in the Fock space norm $\|\cdot\|$, and the vectors in

(14) constitute an orthonormal basis. There is a self-adjoint operator which extends the operator given on $\mathcal{F}_0$ by

$$Q^5 = \sum_{n\geq 0} b_n^\dagger b_n - \sum_{n<0} b_n^\dagger b_n - \sum_{n>0} c_n^\dagger c_n + \sum_{n\leq 0} c_n^\dagger c_n \,.$$

which will also be denoted Q^5; it will be referred to as the *axial* (or chiral) charge operator. Define $\mathcal{F}_0^P \subset \mathcal{F}_0 = \{\text{Ker}\,(Q^5 - 2P)\} \cap \mathcal{F}_0$. The corresponding completions are denoted $\mathcal{H}_0^P$, and are the orthogonal eigenspaces arising in the spectral decomposition of Q^5.

Schwinger regularization

$$Q^{5,reg} = \frac{1}{L} \lim_{\theta \to 0} \sum_{n,n'} \iint \left(b^\dagger_{n'} u^\dagger_{n'} e^{-ik_{n'}y} + c_{n'} v^\dagger_{n'} e^{+ik_{n'}y} \right) e^{ia(y-x)} \\ \times \gamma^5 \left(b_n u_n e^{ik_n x} + c^\dagger_n v_n e^{-ik_n x} \right) \chi_\theta(x-y) \, dy \, dx \, ,$$

and similarly for Hamiltonian. Computation gives:

$$Q^{5,reg} = \sum_{n \geq 0} b^\dagger_n b_n - \sum_{n<0} b^\dagger_n b_n - \sum_{n>0} c^\dagger_n c_n + \sum_{n \leq 0} c^\dagger_n c_n - \frac{aL}{\pi} - 1 \, .$$

Crucial points:

- Requirement of realizing Hamiltonian in some representation where it is semi-bounded leads to introduction of vacuum (“sea-level”) which is not gauge invariant.
- Introduction of non-gauge-invariant vacuum means that Q^5 re-interpreted in associated representation no longer gauge invariant or conserved.
- Schwinger regularization leads to introduction of $Q^{5,reg}$ gauge invariant but still not conserved (evolves in simple way).
- Relatively mild “renormalizations” of Hamiltonian introduced - normal ordering and $e^{i\int a}$; no subtractions of infinities.

Gauge invariant Hamiltonian Schwinger regularization leads to Hamiltonian: $H = H_0 + :H_{coul}:$, where

$$H_0 = -\frac{e^2}{2L}\frac{d^2}{da^2} + \sum_{m\in\mathbb{Z}} |k_m| \left(b_m^\dagger b_m + c_m^\dagger c_m\right) - \frac{a^2 L}{2\pi} - aQ^{5,reg}\,, \tag{15}$$

and

$$H_{coul} = \frac{e^2 L}{2} \sum_{m\neq 0} \frac{1}{k_m^2} \jmath^0(-m)\jmath^0(m) \tag{16}$$

is the Coulomb energy, written in terms of the fourier modes of the current operator

$$j^0 = \sum \jmath^0(m) e^{ik_m x}\,.$$

The symbol $Q^{5,reg}$ indicates the regularized axial charge operator given by:

$$Q^{5,reg} = \sum_{n\geq 0} b_n^\dagger b_n - \sum_{n<0} b_n^\dagger b_n - \sum_{n>0} c_n^\dagger c_n + \sum_{n\leq 0} c_n^\dagger c_n - \frac{aL}{\pi} - 1\,. \tag{17}$$

This expression is also derived from Schwinger regularization; corresponding expression for the regularized ordinary charge is in fact unchanged, i.e.

$$Q = Q^{reg} = \sum_{n\in\mathbb{Z}} \left(b_n^\dagger b_n - c_n^\dagger c_n\right)\,.$$

The total Hilbert space for the theory can now be defined as

$$\mathcal{K} = \{\Psi = \Psi(a) \in \mathcal{H}_0 : \Psi \in L^2([0, \frac{2\pi}{L}]; \mathcal{H}_0)\}\,, \quad (18)$$

with norm defined by $\|\Psi\|^2_{\mathcal{K}} = \int_0^{\frac{2\pi}{L}} \|\Psi\|^2\, da$ where $\| \cdot \|$ is the Fock space norm.

Action of $\mathbb{Z}$ and twisted periodicity

We define a unitary action of the group $\mathbb{Z} = \{g_N(x) = e^{2\pi i N x/L}\}_{N\in\mathbb{Z}}$ of large gauge transformations on $\mathcal{H}_0$. The formulae are best motivated by comparison with the natural expressions in the infinite Dirac sea. There is a unitary operator $\mathbf{\Gamma}$, corresponding to the generator g_1, whose action on the non-interacting vacuum state is

$$\mathbf{\Gamma}\Omega_0 = \Omega_{-1} = b^{\dagger}_{-1} c^{\dagger}_{1} \Omega_0 \,. \tag{19}$$

The action on Fock space is then determined by specifying the action on the set of creation and annihlation operators, on which it acts as a modified shift operator:

$$\begin{aligned} b_n \to \mathbf{\Gamma} b_n \mathbf{\Gamma}^{-1} &= b_{n-1}\,,\ n \neq 0\,, \qquad & b_0 \to \mathbf{\Gamma} b_0 \mathbf{\Gamma}^{-1} &= c^{\dagger}_1 \\ c_n \to \mathbf{\Gamma} c_n \mathbf{\Gamma}^{-1} &= c_{n+1}\,,\ n \neq 0\,, \qquad & c_0 \to \mathbf{\Gamma} c_0 \mathbf{\Gamma}^{-1} &= b^{\dagger}_{-1} \end{aligned}$$

with corresponding relations for the adjoints:

$$\begin{aligned} b^{\dagger}_n \to \mathbf{\Gamma} b^{\dagger}_n \mathbf{\Gamma}^{-1} &= b^{\dagger}_{n-1}\,,\ n \neq 0\,, \qquad & b^{\dagger}_0 \to \mathbf{\Gamma} b^{\dagger}_0 \mathbf{\Gamma}^{-1} &= c_1 \\ c^{\dagger}_n \to \mathbf{\Gamma} c^{\dagger}_n \mathbf{\Gamma}^{-1} &= c^{\dagger}_{n+1}\,,\ n \neq 0\,, \qquad & c^{\dagger}_0 \to \mathbf{\Gamma} c^{\dagger}_0 \mathbf{\Gamma}^{-1} &= b_{-1} \end{aligned} \tag{20}$$

Lemma 4. *These formulae determine an action of $\mathbb{Z}$ on $\mathcal{H}_0$ generated by $\mathbf{\Gamma}$, with the property that $\mathbf{\Gamma}\Omega_P = \Omega_{P-1}$ for all P. Similarly there is a corresponding modified shift action for the inverse $\mathbf{\Gamma}^{-1}$ with the relations inverted, so that in particular $\mathbf{\Gamma}^{-1} \cdot \Omega_0 = b^{\dagger}_0 c^{\dagger}_0 \Omega_0 = \Omega_1$ and more generally $\mathbf{\Gamma}^{-1} \cdot \Omega_P = \Omega_{P+1}$.*

The transformation $\mathbf{\Gamma}$ commutes with Q and so preserves $\mathcal{H}_0$, but it does not commute with Q^5: for example $b_3^\dagger c_2^\dagger b_0^\dagger c_1^\dagger \Omega_0$ is mapped into $b_2^\dagger c_3^\dagger c_2^\dagger b_{-1}^\dagger \Omega_0$, with the eigenvalue of Q^5 reducing by 2. Formally $Q^5 \mathbf{\Gamma}^{-1} = \mathbf{\Gamma}^{-1}(Q^5 - 2)$ on $\mathcal{F}_0$. The interpretation of all these formulae is that *large gauge transformations can create and annihlate fermion/anti-fermion pairs* in a way which seems naively to change the axial charge: an *anomaly*. Nevertheless we have:

Lemma 5. *The Schwinger regularizations of the axial charge and of the Hamiltonian are unchanged by the action of* $\mathbb{Z}$.

Now the gauge transformation g_1 acts on the connection as $a \to a + \frac{2\pi}{L}$, and hence the requirement of gauge invariance means that we should regard the Hamiltonian H as an unbounded operator defined on $\mathcal{K}$ with the following boundary conditions of *twisted periodicity*:

$$\Psi(\frac{2\pi}{L}) = \mathbf{\Gamma}^{-1}\Psi(0) \quad \text{and} \quad \Psi'(\frac{2\pi}{L}) = \mathbf{\Gamma}^{-1}\Psi'(0)\,. \quad (21)$$

(writing prime for $\frac{d}{da}$). A suitable dense domain for the Hamiltonian is $\mathcal{D}$, the space of smooth functions taking values in $\mathcal{F}_0$ which satisfy this twisted periodicity condition, i.e. the restriction to $[0, \frac{2\pi}{L}]$ of the smooth $\mathcal{F}_0$-valued functions which satisfy $\Psi(a + \frac{2\pi}{L}) = \mathbf{\Gamma}^{-1}\Psi(a)$ for all $a \in \mathbf{R}$.

Lemma 6. *$\mathcal{D} \subset \mathcal{K}$ is dense in the norm $\|\cdot\|_{\mathcal{K}}$ on $\mathcal{K}$. The integration by parts formula $\langle \Psi', \Phi \rangle_{\mathcal{K}} = -\langle \Psi, \Phi' \rangle_{\mathcal{K}}$ holds for Ψ, Φ in $\mathcal{D}$.*

Remark 7. *The Fock vacuum Ω_0, thought of as an element of $\mathbb{K}$ which is independent of a, does not satisfy* (21) *and is not gauge invariant. It follows that the interacting (or physical) vacuum cannot be proportional to Ω_0, or indeed any of the unexcited states Ω_P, since $\mathbb{Z}$ maps these states into one another, thus destabilizing the Fock vacuum. The physical vacuum is a linear combination of states of the form $f_P(a)\Omega_P$.*

Theorem 8. *The Hamiltonian is bounded below and essentially self-adjoint on $\mathcal{D} \subset \mathcal{K}$. Vacuum given by:*

$$\Psi_0(a) = \sum_{P\in\mathbb{Z}} f(a - \frac{2\pi}{L}(P - \frac{1}{2}))\Omega_P\,,$$

with

$$f(\tilde{a}) = \frac{L^{\frac{1}{4}}}{\pi^{\frac{3}{8}} e^{\frac{1}{4}}} e^{\frac{-L\tilde{a}^2}{2\sqrt{\pi}e}}$$

and Ω'_P obtained from unexcited states by unitary Bogoliubov transformation. Anomaly equation can be derived as a consequence of the Heisenberg equation of motion.

- Proof uses Lieb-Mattis bosonization (Luttinger model). Gauge invariance couples together the different sectors with $Q^5 = 2P$ as above.
- There is a nontrivial action of $\mathcal{M} = \mathbb{Z}$, the modular group of large gauge transformations, on the non-interacting Fock vacuum Ω_0 given in which maps Ω_0 to the unexcited states Ω_P. The Ω_P are eigenstates of the (unregularized) axial charge operator Q^5 with the following property: no excited fermionic state is occupied which has higher energy than an unoccupied one (amongst states with the correct sign of Q^5).
- It is necessary to take into account this action of $\mathbb{Z}$ in the definition of the Hamiltonian and currents in order to obtain the “correct” gauge invariant expressions. These

expressions, derived using only Schwinger point-splitting regularization, contain terms which give rise to the anomaly in the chiral conservation law.

- Even without “turning on” the Coulomb interaction the interaction between the fermions and the spatial component of the electromagnetic potential a destabilizes the gauge variant non-interacting vacuum Ω_0, producing an interacting vacuum which is gauge invariant (Manton). There is an explicit formula for the interacting vacuum as a linear combination of segments of a gaussian tensored with Ω_P, wrapped around the circle $S^1 = \mathbf{R}/(2\pi/L)$.
- The effect of turning on the Coulomb interaction is to transform the unexcited states Ω_P into dressed versions.

Schwinger Model in External Potential

Introduce external classical potential by adding $\int A_\mu J^{ext,\mu} dxdt$ to the action

$$S = \int \Big[-\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}\Big(i\hbar c \not{D}_A - m \Big)\Psi \Big] \, dx \, dt$$

where J^{ext} is given smooth (for example). The equations of motion are

$$\partial_\mu F^{\mu\nu} = -e \left(\overline{\Psi}\gamma^\nu \Psi \, + \, J^{ext,\nu} \right),$$
$$i\not{D}_A \Psi = \frac{mc}{\hbar}\Psi \, .$$

Can write $A = a + A^{ext}$ and carry out analysis as before.

- Non-autonomous evolution can be constructed in the space $\mathcal{K}$ by a time discretization process.
- Bosonization leads to scalar field coupled to the external field A^{ext} but through the topological coupling term

  $$\int \epsilon^{\mu\nu} A^{ext}_\mu \partial_\nu \Phi \, dxdt$$

  where $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$, rather than usual minimal “covariant derivative” coupling (which doesn’t make sense for real field).

Construction of Evolution Operator

In non-autonomous case need to solve

$$\partial_t \Psi \;=\; A(t)\Psi\,, \qquad \Psi(s) = \Psi_s$$

as $\boxed{\Psi(t) = U(t,s)\Psi_s}$ use Kato's notion of a stable family of generators $\{A(t)\}$ on a Banach space X with norm $\| \, \|$.

Assume either of the following equivalent conditions hold for some positive M and β:

- $\Big\| \prod_{j=1}^N \Big(A(t_j) \,+\, \lambda \Big)^{-1} \Big\| \;\le\; M(\lambda \,-\, \beta)^{-N}$ where here and below $\{t_j\}$ is any nondecreasing finite set of times and$\lambda > \beta$,

- $\Big\| \prod_{j=1}^N \exp[-s_j A(t_j)] \Big\| \le M \exp[+\beta \sum_{j=1}^N s_j]$, for any collection of positive numbers $\{s_j\}$.

Assume existence of dense ctsly embedded subspace

$$Y \subset \mathrm{Dom} A(t) \, \forall t\,,$$

restricted to which each $A(t)$ generates a strongly cts group on Y satisfying $\| \exp[-sA(t)] \| \;\le\; M \exp[s\beta]$. Then the $\{A(t)\}$ generate a strongly cts evolution operator $U(t,s)$ satisfying

$$\|U(t,s)\| \le M \exp[\beta|t-s|]\,, \quad \text{and}$$

$$U(t,r) = U(t,s)U(s,r)\,.$$

TO DO

- Basic excitations of Schwinger model and fermion/ant–fermion bound states. Massless can be thought of as extreme relativistic limit - are these bound states connected to bound states in non-relativistic limit? Need to develop analysis of non-relativistic limit in QFT.

- Non-relativistic limit of $P(\phi)$ The complex $\lambda\phi^4$ field theory in $1+1$ dimensions: prove non-relativistic limit is non-relativistic bosons with point interaction $3\lambda/4m^2\delta(x_1 - x_2$. Partial results only (Dimock) via integral equations (Dyson, Bethe-Salpeter equations).

- Non-relativistic limit of massive Schwinger model Relate to bound states of classical Maxwell-Dirac system. Partially proved Coleman correspondence suggests no fermionic states.

I Classical Nonrelativistic particle

Particle: mass M concentrated at a point

$$\mathbf{X}(t) \in \mathbf{R}^3 \quad \text{at time } t$$

No internal structure.

Newton : if no forces act on a particle it moves at uniform velocity

$$\frac{d^2}{dt^2} = 0$$

Conservation laws:

$$= \frac{d}{dt} \ (momentum)$$

$$E = \frac{2}{2} \ (kinetic\, energy)$$

People used to think that ... when a thing moves it is in a state of motion. This is now known to be a mistake. **Bertrand Russel**

II Quantum Nonrelativistic Particle

The energy momentum relation $E = \frac{2}{2}$ turns into a dispersion relation

$$\frac{1}{\hbar}\,\omega_k = \frac{k^2}{2}$$

for waves

$$\exp[ikx - i\omega_k t]$$

which are the basic solutions of the Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\,\frac{\partial^2\psi}{\partial x^2}$$

with initial data $\psi(x,0) = \psi_0(x)$.

Quantum particle

- still has no internal structure;

- lives in a state characterized e.g. by Fourier transform

$$f(k) = \widehat{\psi}_0(k) \in L^2$$

as

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int f(k)\; \exp[ikx - i\omega_k t]\, dk$$

III Relativistic Particle The relativistic energy momentum relation

$$E^2 = {}^2 + m^2$$

turns into the dispersion relation

$$\omega_k^2 = k^2 + m^2 \qquad (\hbar = 1)$$

and thence the relativistic wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} + m^2 \psi = 0 .$$

Problem of negative energies $E = \pm\sqrt{{}^2 + m^2}$ resolved by saying

- ψ is not a wave function;
- it is a quantum field operator describing creation and annihilation of particles;
- interpretation as multi-particle theory essential.
- ψ is a distribution taking values in space of unbounded operators on a Hilbert space, constrained by Heisenberg relation

$$[\psi(t,x) \, , \, \dot{\psi}(t,y)] = i\delta(x-y) \, ,$$

(*The Reason for Anti-particles* by Richard Feynman.) Leads to three sources of trouble: ultra-violet, infra-red and particle numb

IV Fock space is the (complete) Hilbert direct sum of the symmetric n-fold tensor powers of $L^2(\mathbf{R})$, i.e.

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(L^2(\mathbf{R}))\,.$$

A typical element, $\Psi \in \mathcal{H}$, is a sequence of functions $\{\Psi_n\}_{n=0}^{\infty}$, where $\Psi_n \in L^2(\mathbf{R}^n)$ is symmetric with respect to interchange of any pair of coordinates.

$$\|\Psi\|^2 = \sum \|\Psi_n\|^2_{L^2(\mathbb{R}^n)}\,.$$

The vacuum has $\Psi_0 = 1$ and $\Psi_n = 0$ for $n \geq 1$. Call it Ω or $|\,0\,\rangle$.

Annihilation and creation operators are given, respectively, by

$$(a_k\Psi)_{n-1}(k_1,\ldots,k_{n-1}) = \sqrt{n}\Psi_n(k,k_1,\ldots,k_{n-1})\,,$$

$$(a_k^\dagger\Psi)_{n+1}(k_1,\ldots,k_{n+1}) = \sum_{j=1}^{n+1} \frac{\delta(k-k_j)}{\sqrt{n+1}}\,\Psi_n(k_1,\ldots,\widehat{k_j},\ldots,k_{n+1})\,.$$

(Really define operator valued distributions or quadratic forms.)

V The Free Field

Given dispersion relation $\omega_k = \sqrt{k^2 + 4m^2}$, we define the fields

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int \frac{1}{\sqrt{2\omega_k}} \left(a_k e^{ikx} + a_k^\dagger e^{-ikx} \right) dk \,, \text{ and}$$
$$\pi(x) = \frac{1}{\sqrt{2\pi}} \int -i \sqrt{\frac{\omega_k}{2}} \left(a_k e^{ikx} - a_k^\dagger e^{-ikx} \right) dk \,.$$

Really operator valued distributions

$$\varphi(f) = \int \frac{1}{\sqrt{2\omega_k}} \left(a_k \, \widehat{f}(-k) + a_k^\dagger \, \widehat{f}(k) \right) dk \,,$$

where $\widehat{f}(k) = (2\pi)^{-1/2} \int e^{-ikx} f(x) \, dx \in \mathcal{S}(\mathbf{R})$ is the Fourier transform.

Notice vacuum expectation infinite:

$$\langle 0 | \varphi(x)^2 | 0 \rangle = \| \varphi(x) \Omega \|^2 = \frac{1}{4\pi} \int \frac{dk}{\omega_k} = +\infty \,.$$

Wick ordering - move annihilation operators to right - gives

$$\langle 0 : | \varphi(x)^2 : | 0 \rangle = 0 \,.$$

Physically : removes self interaction of particles on themselves.

VI Regularized fields Let $\delta_1 \in C_0^\infty(\mathbf{R})$ be a nonnegative, even function with $\delta_1(x) = 0$ for $|x| \geq 1$, and satisfying $\int \delta_1(x)\, dx = 1$. For $\kappa > 0$ define $\delta_\kappa(x) = \kappa\delta_1(\kappa x)$, so that the operator $\delta_\kappa *$ is an approximation to the identity. Regularized fields:

$$\varphi_\kappa(x) = \int \frac{\chi_\kappa(k)}{\sqrt{2\omega_k}} \left(a_k e^{ikx} + a_k^\dagger e^{-ikx}\right) dk\,,$$
$$\pi_\kappa(x) = \int -i\chi_\kappa(k) \sqrt{\frac{\omega_k}{2}} \left(a_k e^{ikx} - a_k^\dagger e^{-ikx}\right) dk\,,$$

where $\chi_\kappa(k) = \chi(k/\kappa)$ with $\chi(k) = \widehat{\delta}_1(k)$.

Regularization amounts to a smooth momentum cut-off at scales large compared to κ since $\chi_\kappa(k) = \widehat{\delta}_1(k/\kappa)$:

$$\gamma_\kappa = \langle 0 | \varphi_\kappa(x)^2 | 0 \rangle = \int \frac{|\chi_\kappa(k)|^2\, dk}{2\omega_k} < +\infty\,.$$

But Wick ordering interferes with boundedness below:

$$\begin{aligned} :\varphi_\kappa(x)^4: &= \varphi_\kappa(x)^4 - 6\gamma_\kappa\varphi_\kappa^2 + 3\gamma_\kappa^2 \\ &= (\varphi_\kappa - 3\gamma_\kappa)^2 - 6\gamma_\kappa^2 \\ &\geq -6\gamma_\kappa^2 \end{aligned}$$

so the pointwise lower bound diverges as cut-off removed.

Wick Operators Given a function or distribution $w \in \mathcal{S}(\mathbf{R}^{m+n})$, written$w = w(\underline{k}, \underline{k}')$ for $\underline{k} = (k_1, \ldots k_m)$ and $\underline{k}' = (k_1', \ldots k_n')$, the Wick operator on Fock space is given by

$$w = \int_{\mathbf{R}^{n+m}} a^\dagger(k_m) \ldots a^\dagger(k_1) w(\underline{k}, \underline{k}') \\ \times a(k_1') \ldots a(k_n') d\underline{k} d\underline{k}' .$$

Here $d\underline{k}' = \prod_{j=1}^{n} dk_j'$ and $d\underline{k} = \prod_{j=1}^{m} dk_j$.

Writing $= \int a^\dagger(k) a(k)\, dk$ for the number operator as usual, we have the following bounds in the case that the kernel is square integrable:

$$\|(\mathbb{1}+)_w^{-m/2}(\mathbb{1}+)^{-n/2}\| \leq \|w\|$$

and, more generally for $a + b \geq m + n$,

$$\|(\mathbb{1}+)^{-a/2}{}_w(\mathbb{1}+)^{-b/2}\| \\ \leq (1 + |m-n|^{|m-n|/2})\|w\|$$

where on the left hand side $\| \cdot \|$ means Fock space operator norm, while on the right hand side $\|w\|$ means the norm of the kernel w as an operator $\mathsf{Sym}^n(L^2(\mathbf{R})) \to \mathsf{Sym}^m(L^2(\mathbf{R}))$.

The Wick polynomial $\int : \varphi_\kappa(x)^4 : b(x)\,dx$ determined by a regularized field and a spatial cut-off $b \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ determines a Wick operator (with obvious conventions for $j = 0, 4$):

$$\sum_{j=0}^{4} \binom{4}{j} \int_{\mathbf{R}^4} a^\dagger(k_1) \ldots a^\dagger(k_j) w(\underline{k}, \underline{k}') \\ \times a(-k_{j+1}) \ldots a(-k_4) dk_1 \ldots dk_4$$

where

$$v(k) = \widehat{b}(-\sum_{j=1}^{4} k_j) \prod_{j=1}^{4} \frac{\chi_\kappa(k_j)}{2\omega_{k_j}} \in \mathcal{S}'(\mathbf{R}^4)\,.$$

The preceding Wick operator bounds applied to this give for any $\epsilon > 0$ a number C_ϵ such that

$$\left\| \frac{\int : \varphi_\kappa(x)^4 : b(x)\,dx - \int : \varphi_{\kappa'}(x)^4 : b(x)\,dx}{(\mathbb{1} +)^4} \right\| \\ \leq \frac{C_\epsilon}{(\min\{\kappa, \kappa'\})^{\frac{1}{2} - \epsilon}}$$

VII Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial x^2} + V\psi$$

with initial data $\psi(x,0) = \psi_0(x)$.

Feynman (PhD thesis, 1942) reformulated quantum mechanics :

$$\psi(x,t) = \int \text{exp}\Big\{\frac{i}{\hbar}\int_0^t\Big(\frac{1}{2}\dot{X}^2 - V(X(s))\Big)ds\Big\} \times \psi_0(X(t)) \quad \boxed{\prod_{0\leq s\leq t} dX(s)}$$

in terms of “complex probability amplitudes” by summing over paths with $X(0) = x$. Put $\hbar = 1$ from now on.

Mathematical analysis of semi-group via the Feynman-Kac formula: after Wick rotation

$$\boxed{t \to -it}$$

to Euclidean time $\text{exp}[-tH]\psi_0(x)$ is given by

$$\begin{aligned} &{}_x\Big[\text{exp}\{-\int_0^t V(X(s))ds\Big\}\,\psi_0(X(t))\Big] \\ &= \int \text{exp}\Big\{-\frac{1}{\hbar}\int_0^t V(X(s))\,ds\Big\}\,\psi_0(X(t))\,d_x(X) \end{aligned}$$

(expectation w.r.t. Wiener measure d_x on paths starting at $x = X(0)$.)

VIII Rewrite Feynman-Kac formula as

$$\left(F\,,\,e^{-tH}\,G\right)_{L^2}\;=\;\int\,_x\left(F(x)\,e^{-\int_0^t J_sV\,ds}J_tG\right)\,dx$$

where $J_sV\,:\,C(\mathbf{R})\to\mathbf{R}$ is the function on path space $X\mapsto V(X(s))$ etc.

X: Gaussian process with covariance $\frac{1}{2}\min\{s,t\}$, i.e. evolution is obtained by averaging over all Brownian paths with diffusion $\frac{1}{2}$.

For an oscillator

$$i\frac{\partial\psi}{\partial t}\;=\;-\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}+\frac{1}{2}\omega^2x^2\psi+V\psi$$

the formula generalizes via introduction of the oscillator process defined as the Gaussian process indexed by $t\in\mathbf{R}$ with covariance

$$(q(t)q(s))\;=\;\frac{e^{-\omega|t-s|}}{2\omega}$$

Averaging over oscillator process we can write $\psi=e^{-tH}\psi_0$ where

$$(F\,,\,\exp[-tH]G)\;=\;\left(J_0F\,e^{-\int_0^t J_sV\,ds}J_tG\right)$$

where again $J_sV\,:\,C(\mathbf{R})\to\mathbf{R}$ is the function on path space with value $V(q(s))$.

IX Classical Harmonic Oscillator: Hamiltonian

$$H_{osc} = \frac{1}{2}\left(p^2 + \omega^2 x^2\right)$$

Classical free field

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + 4m^2\phi = 0$$

we have, with $\pi = \partial_t\phi = \dot{\phi}$, the Hamiltonian

$$H = \frac{1}{2}\int \left(\pi^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + 4m^2\phi^2\right) dx$$

or, with, $\widehat{\phi}(k) = (2\pi)^{-\frac{1}{2}}\int e^{-ikx}\phi(x)dx$ etc

$$H = \frac{1}{2}\int \left(|\widehat{\pi}(k)|^2 + \left(k^2 + 4m^2\right)|\widehat{\phi}(k)|^2 \right) dk$$

Free field: an infinite collection of oscillators of frequency $\omega_k = \sqrt{k^2 + 4m^2}$.

Nelson: used this to generalize Feynman-Kac to quantum fields, to describe semi-group e^{-tH} acting on the Fock space.

X Feynman-Kac-Nelson Formula We need two facts

- $\exists$ a Gaussian measure γ on $\mathcal{S}'(\mathbf{R})$ giving a model of Fock space (Schrödinger representation) such that

$$= \overline{\bigoplus_n} \text{Sym}^n L^2(\mathbf{R}) = L^2\left(\mathcal{S}'(\mathbf{R}), d\gamma\right)$$

$$\begin{aligned}(\phi(f)\phi(g)) &= \int \phi(f)\phi(g)\, \gamma(d\phi) \\ &= \int \frac{\overline{f(k)} g(k)}{2\omega_k}\, dk\,.\end{aligned}$$

- $\exists$ a Gaussian measure μ on $\mathcal{S}'(\mathbf{R}^2)$ such that

$$\begin{aligned}(\Phi(f)\Phi(g)) &= \int \Phi(f)\Phi(g)\, \mu(d\Phi) \\ &= \iiint \frac{\overline{f(s,k)} e^{-|t-s|\omega_k} g(t,k)}{2\omega_k}\, dk ds dt\,.\end{aligned}$$

$$\boxed{\left(F\,,\, e^{-tH}\, G\right)_{L^2(\gamma)} = \left(J_0 F\, e^{-\int_0^t J_s V ds} J_t G\right)}$$

Here ϕ is the spatial Fourier transform of Euclidean field

$$\phi(t,k) = (2\pi)^{-\frac{1}{2}} \int e^{-ikx}(t,x)dx$$

i.e. arguments are time t and spatial Fourier variable k. The Euclidean field is Gaussian process on $\mathcal{S}'(\mathbf{R}^2)$ with covariance

$$\Big(\Phi_E(f)\Phi_E(g)\Big) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\overline{\hat{f}(k)}\hat{g}(k)}{k^2 + 4m^2}\, dk\,.$$

At each time t there exists an isometry $J_t : L^2(d\gamma) \to L^2(d\mu)$ given on Wick monomials by

$$J_t : \phi(f)^n : \to : \phi(t,f)^n :$$

Wick monomials obtained by orthogonalization process with respect to the corresponding Gaussian measure. They generate polynomials which are dense in the corresponding L^2 space.

XI Glimm-Jaffe PSC Expansion

Introduce an overall large upper momentum cut-off κ, and sequence

$$\kappa_1 < \kappa_2 < \kappa_3 < \cdots < \kappa_{n-1} < \kappa \leq \kappa_n \qquad \kappa_n = e^{\sqrt{\nu}}$$

and corresponding cut-off Hamiltonians $h_\nu = H^{\kappa_\nu}$ for $1 \leq \nu \leq n-1$, and then and $h_n = H^\kappa$ if $\nu \geq n$. Want bounds independent of κ or equivalently n.

Iterated Duhamel:

$$\begin{aligned} e^{-tH^\kappa} = \; & e^{-th_1} - \int_0^t e^{-(t-s_1)h_2}(H^\kappa - h_1)e^{-s_1 h_1}\, ds_1 \\ & - \int_0^t \int_{s_1}^t e^{-(t-s_2)h_3}(H^\kappa - h_2)e^{-(s_2-s_1)h_2} \\ & \qquad \times (H^\kappa - h_1)e^{-s_1 h_1}\, ds_2 ds_1 \\ & \cdots \\ & - (-1)^n \int_0^t \cdots \int_{s_{n-2}}^t e^{-(t-s_{n-1})H^\kappa}(H^\kappa - h_{n-1}) \\ & \qquad \times \prod_{\nu=2}^{n-1} \left(e^{-(s_\nu - s_{\nu-1})h_\nu}(H^\kappa - h_{\nu-1}) \right) \\ & \qquad \times e^{-s_1 h_1} \prod_{j=1}^{n-1} ds_j \,. \end{aligned}$$

The aim is to prove an *operator* lower bound $H^\kappa \geq -c_0 > -\infty$ which is uniform in κ, in spite of the fact that *pointwise* H_I^κ is not uniformly bounded below. Indeed normal ordering gives

$$\begin{aligned} :\varphi_\kappa(x)^4: &= \varphi_\kappa(x)^4 - 6\varphi_\kappa(x)^2\gamma_\kappa + 3\gamma_\kappa^2 \\ &\geq -6\gamma_\kappa^2 \end{aligned}$$