SINGULAR PERTURBATIONS OF DIRAC HAMILTONIANS: selfadjointness and spectrum.

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The Operator

- $\partial_t \psi \;=\; iH\psi \quad ; \quad H \;=\; H_0 + \mathbb{V} \quad , \quad \psi \;=\; \psi(x,t) \quad , \quad \mathbb{V}(x)$ hermitian
- $H_0 = \frac{1}{i}\alpha \cdot \nabla + m\beta$
- $H_0^2 = -\Delta + m^2$

$$\begin{aligned}
&\alpha \cdot \alpha = \mathbb{1} \qquad \alpha = (\alpha_j) \\
&\alpha\beta + \alpha\beta = 0 \\
&\alpha_j \alpha_k + \alpha_k \alpha_j = 0 \qquad j \neq k \quad ; \quad \alpha_j^2 = \mathbb{1} \qquad j = 1, 2, 3
\end{aligned}$$

- If $x \in \mathbb{R}^3$ then $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$, $\phi, \chi \in \mathbb{C}^2$ (spinors).
- $\mathbb{V}$: "critical" $\qquad \frac{1}{\lambda}\mathbb{V}\left(\frac{x}{\lambda}\right) \sim \mathbb{V}(x)$

Example: Coulomb $\qquad \mathbb{V} = \frac{-\lambda}{|x|}\mathbb{1}$

General Questions

(a) Self–adjointness.

(b) Spectrum: Characterization of the ground state by the "right inequality".

Similar questions for a non linear $\mathbb{V}$ always assume some smallness condition on $\mathbb{V}$.

(c) What is a small/big perturbation of H_0?

Coulomb Potential

- $H_0 - \dfrac{\lambda}{|x|}$

(a) Self–adjointness: **Rellich '53**, **Schminke '72**, **Wust '75**, **Nenciu '76**, **Kato '80– '83** (Kato–Nenciu inequality)

Final answer: $|\lambda| < 1$.

(b) "Ground state" $(\lambda > 0)$ Minimization process (**Dolbeault**, **Esteban**, **Séré '00**):

- Variational inequality for $\phi \left(\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \right)$.
- Hardy–Kato–Nenciu type inequalities (**Dolbeault**, **Duoandikoetxea**, **Esteban**, **Loss**, **V. '00**).

Recall Birman–Schwinger principle:

$$\frac{d}{da}\left(\frac{1}{\lambda(a)}\right) \sim \langle (H-a)^{-2} g_a, g_a \rangle = \|(H-a)^{-1} g_a\|^2 \geq 0$$

(assume g_a independent of a)

This suggests another way of obtaining the ground state for the Coulomb potential $V(x) = -\frac{\lambda}{|x|}$:

$$\frac{m^2 - a^2}{m^2} \int \frac{|\psi|^2}{|x|} \leq \int \left| \left(\frac{1}{i} \alpha \cdot \nabla + m\beta + a \right) \psi \right|^2 |x|$$

(**Arrizabalaga**, **Duoandikoetxea**, **V. '13**; **Cassano**, **Pizzichilo**, **V. '17**)

The inequality is optimal and it is achieved for $A > 0$ by the ground state of $V_a(x) = -\frac{m^2 - a^2}{m^2} \frac{1}{|x|}$.

The proof is a consequence of the "uncertainty principle".

- $2\mathrm{Re}\,\langle S\psi, A\psi\rangle = \langle (SA - AS)\psi, \psi\rangle$ if $S^* = S$ and $A^* = -A$.

- $2\mathrm{Re}\,\langle A_1\psi, A_2\psi\rangle = -\langle (A_1A_2 + A_2A_1)\psi, \psi\rangle$ if $A_1^* = -A_1$ and $A_2^* = -A_2$.

In our case the right choice is:

$$2\mathrm{Re}\,\langle \underbrace{(\alpha\cdot\nabla + i(m\beta + a))}_{A_1}\,\psi, (1 + \sigma\cdot L)\mathbb{1}\,\alpha\cdot\frac{x}{|x|}\left(\underbrace{\frac{a}{m}\beta}_{S} + \underbrace{1}_{A_2}\right)\rangle.$$

Electrostatic Shell Interactions:

$\Omega \subset \mathbb{R}^3$ bounded smooth domain

$\sigma =$ surface measure on $\partial\Omega$

$N =$ outward unit normal vector field on $\partial\Omega$

Electrostatic shell potential $V_\lambda = \lambda\delta_{\partial\Omega}$**:**

$$\lambda \in \mathbb{R}, \qquad V_\lambda(\varphi) = \frac{\lambda}{2}(\varphi_+ + \varphi_-)$$

$\varphi_\pm =$ non–tangential boundary values of $\varphi : \mathbb{R}^3 \to \mathbb{C}^4$ when approaching from Ω or $\mathbb{R}^3 \setminus \overline{\Omega}$

Electrostatic shell interaction for H**:** $H + V_\lambda$

(a) Self-Adjointess

If $\lambda \neq \pm 2 \quad \Longrightarrow \quad H + V_\lambda$ is self-adjoint on $\mathcal{D}(H + V_\lambda)$.

(**[Arrizabalaga, Mas, V., 2014]**,
more general **[Posilicano, 2008]**
Ω ball $\longrightarrow$ **[Dittrich, Exner, Seba, 1989]**)

$a \in (-m, m)$

$$\phi^a(x) = \frac{e^{-\sqrt{m^2-a^2}\,|x|}}{4\pi|x|}\left[a + m\beta + \left(1 - \sqrt{m^2-a^2}\,|x|\right) i\alpha \cdot \frac{x}{|x|^2}\right]$$

$$= \text{ fundamental solution of } H - a$$

$$\mathcal{D}(H + V_\lambda) = \Big\{\varphi : \quad \varphi = \phi^0 * (Gdx + gd\sigma),\ G \in L^2((R)^3)^4 \ \ g \in L^2(\partial\Omega)^4,$$

$$\lambda\left(\phi^0 * (Gdx)\right)\big|_{\partial\Omega} = -\left(1 + \lambda C^0_{\partial\Omega}\right) g)\Big\}$$

where $C^a_{\partial\Omega}(g)(x) = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} \phi^a(x-y)g(y)d\sigma(y)$, $x \in \partial\Omega$.

(b) Point Spectrum on $(-m,m)$ for $H+V_\lambda$

Birman–Schwinger principle: $a \in (-m,m), \quad \lambda \in \mathbb{R} \setminus \{0\},$

$$\ker(H+V_\lambda - a) \neq 0 \quad \Longleftrightarrow \quad \ker\left(\frac{1}{\lambda} + C^a_{\partial\Omega}\right) \neq 0$$

(problem in $\mathbb{R}^3$) **(problem in $\partial\Omega$)**

Properties of $C^a_{\partial\Omega}$, $\quad a \in [-m,m]$:

(a) $C^a_{\partial\Omega}$ bounded self–adjoint operator in $L^2(\partial\Omega)^4$.

(b) $\left[C^a_{\partial\Omega}(\alpha \cdot N)\right]^2 = -\frac{1}{4} I_d.$ $\quad \left(\alpha \cdot N = \sum_{j=1}^{3} \alpha_j N_j \text{ multiplication operator}\right)$

$$\ker\left(\frac{1}{\lambda} + C^a_{\partial\Omega}\right) \neq 0 \quad \begin{cases} \overset{(a)}{\Longrightarrow} \quad |\lambda| \geq \lambda_l(\partial\Omega) > 0 \quad \text{and} \quad \lambda_l(\partial\Omega) \leq 2 \\ \overset{(b)}{\Longrightarrow} \quad |\lambda| \leq \lambda_u(\partial\Omega) < +\infty \quad \text{and} \quad \lambda_u(\partial\Omega) \geq 2 \end{cases}$$

Therefore, $\ker(H+V_\lambda - a) \neq 0 \quad \Longrightarrow \quad |\lambda| \in [\lambda_l(\partial\Omega), \lambda_u(\partial\Omega)]$

Theorem [AMV2016].– $\Omega \subset \mathbb{R}^3$ bounded smooth domain. If

$$m\frac{\text{Area}(\partial\Omega)}{\text{Cap}(\overline{\Omega})} > \frac{1}{4\sqrt{2}},$$

then

$$\sup\{|\lambda| : \ker(H + V_\lambda - a) \neq 0 \text{ for some } a \in (-m, m)\} \\ \geq 4\left(m\frac{\text{Area}(\partial\Omega)}{\text{Cap}(\overline{\Omega})} + \sqrt{m^2\frac{\text{Area}(\partial\Omega)^2}{\text{Cap}(\overline{\Omega})^2} + \frac{1}{4}}\right)$$

and

$$\inf\{|\lambda| : \ker(H + V_\lambda - a) \neq 0 \text{ for some } a \in (-m, m)\} \\ \leq 4\left(-m\frac{\text{Area}(\partial\Omega)}{\text{Cap}(\overline{\Omega})} + \sqrt{m^2\frac{\text{Area}(\partial\Omega)^2}{\text{Cap}(\overline{\Omega})^2} + \frac{1}{4}}\right)$$

In both cases, $=$ holds $\iff$ Ω is a ball.

Joint work with T. Ourmieres-Bonafos.

Recent work by

-Benguria, Fournais, Stockmeyer, Van den Bosch
-Behrndt, Exner, Holzmann, Lotoreichik
-Behrndt, Holzmann

For $\lambda \in \mathbb{R}$, we introduce the matrix valued function:

$$\mathcal{P}_\lambda = \frac{\lambda}{2} + i(\alpha \cdot \mathbf{n}).$$

For $(u_+, u_-) \in H^1(\Omega_+)^4 \times H^1(\Omega_-)^4$ we define the following transmission condition in $H^{1/2}(\partial\Omega)^4$

$$(*) \qquad \mathcal{P}_\lambda t_{\partial\Omega} u_+ + \mathcal{P}_\lambda^* t_{\partial\Omega} u_- = 0, \quad \text{on } \partial\Omega.$$

Alternativaley, as $\mathcal{P}_\lambda$ is invertible, we can see the transmission condition as

$$t_{\partial\Omega} u_+ = \mathcal{R}_\lambda t_{\partial\Omega} u_-, \text{ with } \mathcal{R}_\lambda := \frac{1}{\lambda^2/4+1}\Big(1 - \frac{\lambda^2}{4} + \lambda(i\alpha \cdot \mathbf{n})\Big).$$

Definition.– Let $\lambda \in \mathbb{R}$ and $m \in \mathbb{R}$. The Dirac operator coupled with an electrostatic δ–shell interaction of strength λ is the operator $\Big(\mathcal{H}_\lambda(m), \mathrm{dom}\,(\mathcal{H}_\lambda(m))\Big)$, acting on $L^2(\mathbb{R}^3)^4$ and defined on the domain

$$\mathrm{dom}\,\Big(\mathcal{H}_\lambda(m)\Big) = \Big\{(u_+, u_-) \in H^1(\Omega_+)^4 \times H^1(\Omega_-)^4 \;:\; (u_+, u_-) \text{ satisfies } (*)\Big\}$$

It acts in the sense of distributions as $\mathcal{H}_\lambda(m)u = \Big(\mathcal{H}(m)u_+, \mathcal{H}(m)u_-\Big)$ where we identify an element of $L^2(\Omega_+)^4 \times L^2(\Omega_-)^4$ with an element of $L^2(\mathbb{R}^3)^4$.

Theorem.– Let $m \in \mathbb{R}$. The following holds:

(i) If $\lambda \neq \pm 2$, the operator $\Big(\mathcal{H}_\lambda(m), \mathrm{dom}\,\big(\mathcal{H}_\lambda(m)\big)\Big)$ is self-adjoint.

(ii) If $\lambda = \pm 2$, the operator $\Big(\mathcal{H}_\lambda(m), \mathrm{dom}\,\big(\mathcal{H}_\lambda(m)\big)\Big)$ is essentially self-adjoint and we have

$$\mathrm{dom}\,(\mathcal{H}_\lambda(m)) \subsetneq \mathrm{dom}\,(\overline{\mathcal{H}}_\lambda(m)) = \Big\{(u_+, u_-) \in H(\alpha, \Omega_+) \times H(\alpha, \Omega_-) : (u_+, u_-) \text{ satisfies } (*)\Big\},$$

where the transmission condition holds in $H^{-1/2}(\partial\Omega)^4$.

Here:

- $H(\alpha,\Omega) \quad := \quad \Big\{u \in L^2(\Omega)^4 : \mathcal{H}u \in L^2(\Omega)^4\Big\} =$

$$\Big\{u \in L^2(\Omega)^4 : (\alpha\cdot\mathbf{D})u \in L^2(\Omega)^4\Big\},$$

- $\alpha\cdot\mathbf{D} = \dfrac{1}{i}\alpha\cdot\nabla.$

Let $\varepsilon = \pm 1$ and $\lambda = 2\varepsilon$. Let $u = (u_+, u_-) \in \mathrm{dom}\big(\mathcal{H}_\lambda(m)\big)$, $u_\pm$ can be rewritten $u_\pm = (u_\pm^{[1]}, u_\pm^{[2]})$ and, for $x \in \partial\Omega$, the transmission condition reads

$$\begin{pmatrix} u_+^{[1]}(x) \\ u_+^{[2]}(x) \end{pmatrix} = \begin{pmatrix} 0 & -i\varepsilon\sigma\cdot\mathbf{n}(x) \\ -i\varepsilon\sigma\cdot\mathbf{n}(x) & 0 \end{pmatrix}\begin{pmatrix} u_-^{[1]}(x) \\ u_-^{[2]}(x) \end{pmatrix}$$

$$= \begin{pmatrix} -i\varepsilon\sigma\cdot\mathbf{n}u_-^{[2]}(x) \\ -i\varepsilon\sigma\cdot\mathbf{n}u_-^{[1]}(x) \end{pmatrix}.$$

For $u \in H^1(\mathbb{R}^3 \setminus \Omega)^4$, $\delta_{\partial\Omega} u$ is the distribution defined as

$$\langle \delta_{\partial\Omega} u, v \rangle := \frac{1}{2} \int_{\partial\Omega} \langle t_{\partial\Omega} u_+(x) + t_{\partial\Omega} u_-(x), v(x) \rangle_{\mathbb{C}^4} ds(x),$$

for all $v \in \mathcal{C}_0^\infty(\mathbb{R}^3)^4$.

We are interested in functions $u \in L^2(\mathbb{R}^3)^4$ such that

$$\big(\mathcal{H}(m) + \lambda \delta_{\partial\Omega}(x) \mathrm{Id}\big) u \in L^2(\mathbb{R}^3)^4.$$

For example, if $u = (u_+, u_-) \in H^1(\Omega_+)^4 \times H^1(\Omega_-)^4$, a computation in the sense of distributions yields

$$\begin{aligned} \big(\mathcal{H}(m) + \lambda \delta_{\partial\Omega}(x) \mathrm{Id}\big) u \quad &= \alpha \cdot \mathbf{D} u + m\beta u + \frac{\lambda}{2} \left(t_{\partial\Omega} u_+ + t_{\partial\Omega} u_- \right) \delta_{\partial\Omega} \\ = \{(\alpha \cdot \mathbf{D}) u\} + m\beta u - i\alpha \cdot \mathbf{n} \left(t_{\partial\Omega} u_+ + t_{\partial\Omega} u_- \right) \delta_{\partial\Omega} &+ \frac{\lambda}{2} \left(t_{\partial\Omega} u_+ + t_{\partial\Omega} u_- \right) \delta_{\partial\Omega} \\ = \underbrace{\{(\alpha \cdot \mathbf{D}) u\} + m\beta u}_{\in L^2(\mathbb{R}^3)^4} + \bigg(\frac{\lambda}{2} (t_{\partial\Omega} u_+ + t_{\partial\Omega} u_-) &- i\alpha \cdot \mathbf{n} (t_{\partial\Omega} u_- - t_{\partial\Omega} u_+) \bigg) \delta_{\partial\Omega}, \end{aligned}$$

where we set

$$\{(\alpha \cdot \mathbf{D})u\}|_{\Omega_\pm} = (\alpha \cdot \mathbf{D})u_\pm.$$

Now, we would like the last term in the right-hand side to be zero. It reads:

$$\left(\frac{\lambda}{2}\mathrm{Id} + i\alpha \cdot \mathbf{n}\right) t_{\partial\Omega} u_+ + \left(\frac{\lambda}{2}\mathrm{Id} - i\alpha \cdot \mathbf{n}\right) t_{\partial\Omega} u_- = 0.$$

In particular, it justifies that for $u \in \operatorname{dom}\big(\mathcal{H}_\lambda(m)\big)$, $\mathcal{H}_\lambda(m)u \in L^2(\mathbb{R}^3)^4$.

Proposition.– The trace operator $t_{\partial\Omega}$ extends into a continuous map $t_{\partial\Omega} : H(\alpha, \Omega) \to H^{-1/2}(\partial\Omega)^4$.

Theorem.– ϕ extends into a bounded operator from $H^{-1/2}(\partial\Omega)^p$ to $H(\alpha,\Omega)$.

The boundary integral operator is defined taking the boundary data of ϕ on $\partial\Omega$

$$C(g) = t_{\partial\Omega}\big(\phi(g)\big).$$

An important consequence:

Corollary.– The following operator is continuous:

$$C : H^{-1/2}(\partial\Omega)^4 \to H^{-1/2}(\partial\Omega)^4,$$

Proposition.– Let $u \in H(\alpha,\Omega)$. Assume that $t_{\partial\Omega}u \in H^{1/2}(\partial\Omega)^4$, then we have $u \in H^1(\Omega)^4$.

The Calderón projectors are the bounded linear operators from $H^{-1/2}(\partial\Omega)^4$ onto itself defined as:

$$\mathcal{C}_\pm = \pm i C_\pm(\alpha \cdot \mathbf{n}).$$

As $\partial\Omega$ is $\mathcal{C}^2$, the multiplication by $\alpha \cdot \mathbf{n}$ is a bounded linear operator from $H^{-1/2}(\partial\Omega)^4$ onto itself. Thus the definition makes sense.

Their formal adjoints are:

$$\mathcal{C}_\pm^* = \mp i(\alpha \cdot \mathbf{n}) C_\mp.$$

By definition, $\mathcal{C}_\pm^*$ is a linear bounded operator from $H^{-1/2}(\partial\Omega)^4$ onto itself.

Proposition.– We have:

(i) $\mathcal{C}'_\pm = \mathcal{C}^*_\pm|_{H^{1/2}(\partial\Omega)^4}$ and $(\mathcal{C}^*_\pm)' = \mathcal{C}_\pm|_{H^{1/2}(\partial\Omega)^4}$. In particular $\mathcal{C}_\pm|_{H^{1/2}(\partial\Omega)^4}$ and $\mathcal{C}^*_\pm|_{H^{1/2}(\partial\Omega)^4}$ are bounded operators from $H^{1/2}(\partial\Omega)^4$ onto itself,

(ii) $(\mathcal{C}_\pm)^2 = \mathcal{C}_\pm$ and $(\mathcal{C}^*_\pm)^2 = \mathcal{C}^*_\pm$,

(iii) $\mathcal{C}_+ + \mathcal{C}_- = \mathrm{Id}$ and $\mathcal{C}^*_+ + \mathcal{C}^*_- = \mathrm{Id}$,

(iv) $(\alpha \cdot \mathbf{n})\mathcal{C}_\pm = \mathcal{C}^*_\mp(\alpha \cdot \mathbf{n})$ and $\mathcal{C}_\pm(\alpha \cdot \mathbf{n}) = (\alpha \cdot \mathbf{n})\mathcal{C}^*_\mp$.

Note that the Calderón projectors satisfy:

$$\mathcal{C}_\pm - \mathcal{C}_\pm^* = \pm i\mathcal{A},$$

where $\mathcal{A}$ does not depend on the sign $\pm$. Roughly speaking, $\mathcal{A}$ measures the defect of self-adjointness of the Calderón projectors.

Proposition.– The operator $\mathcal{A}$ extends into a bounded operator from $H^{-1/2}(\partial\Omega)^4$ to $H^{1/2}(\partial\Omega)^4$ and it is compact.

Self-adjointness

Let $u \in \operatorname{dom}\big(\mathcal{H}_\lambda(m)^*\big)$

$$(**) \qquad \mathcal{P}_\lambda t_{\partial\Omega} u_+ + \mathcal{P}_\lambda^* t_{\partial\Omega} u_- = 0.$$

We have:

$$(**) \iff \begin{cases} \mathcal{C}_+\big(\mathcal{P}_\lambda t_{\partial\Omega} u_+ + \mathcal{P}_\lambda^* t_{\partial\Omega} u_-\big) & = & 0 \\ \mathcal{C}_-\big(\mathcal{P}_\lambda t_{\partial\Omega} u_+ + \mathcal{P}_\lambda^* t_{\partial\Omega} u_-\big) & = & 0 \end{cases},$$

$$\iff \begin{cases} \frac{\lambda}{2}\Big(\mathcal{C}_+(t_{\partial\Omega} u_+) + \mathcal{C}_+(t_{\partial\Omega} u_-)\Big) + i(\alpha\cdot\mathbf{n})\mathcal{C}_-^*(t_{\partial\Omega} u_+ - t_{\partial\Omega} u_-) & = & 0 \\ \\ \frac{\lambda}{2}\Big(\mathcal{C}_-(t_{\partial\Omega} u_+) + \mathcal{C}_-(t_{\partial\Omega} u_-)\Big) + i(\alpha\cdot\mathbf{n})\mathcal{C}_+^*(t_{\partial\Omega} u_+ - t_{\partial\Omega} u_-) & = & 0 \end{cases},$$

$$\iff \begin{cases} \frac{\lambda}{2}\Big(\mathcal{C}_+(t_{\partial\Omega} u_+) + \mathcal{C}_+(t_{\partial\Omega} u_-)\Big) + i(\alpha\cdot\mathbf{n})\Big(\mathcal{C}_-(t_{\partial\Omega} u_+) - \mathcal{C}_-(t_{\partial\Omega} u_-) + i\mathcal{A}(t_{\partial\Omega} u_+ - t_{\partial\Omega} u_-)\Big) & = & 0 \\ \\ \frac{\lambda}{2}\Big(\mathcal{C}_-(t_{\partial\Omega} u_+) + \mathcal{C}_-(t_{\partial\Omega} u_-)\Big) + i(\alpha\cdot\mathbf{n})\Big(\mathcal{C}_+(t_{\partial\Omega} u_+) - \mathcal{C}_+(t_{\partial\Omega} u_-) - i\mathcal{A}(t_{\partial\Omega} u_+ - t_{\partial\Omega} u_-)\Big) & = & 0 \end{cases}.$$

This system rewrites as:

$$\begin{pmatrix} \frac{\lambda}{2} & -i\alpha\cdot\mathbf{n} \\ i\alpha\cdot\mathbf{n} & \frac{\lambda}{2} \end{pmatrix} \begin{pmatrix} \mathcal{C}_+(t_{\partial\Omega}u_+) \\ \mathcal{C}_-(t_{\partial\Omega}u_-) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\lambda}{2} & -i\alpha\cdot\mathbf{n} \\ i\alpha\cdot\mathbf{n} & -\frac{\lambda}{2} \end{pmatrix} \begin{pmatrix} \mathcal{C}_+(t_{\partial\Omega}u_-) \\ \mathcal{C}_-(t_{\partial\Omega}u_+) \end{pmatrix} + \begin{pmatrix} (\alpha\cdot\mathbf{n})\mathcal{A}(t_{\partial\Omega}u_+ - t_{\partial\Omega}u_-) \\ -(\alpha\cdot\mathbf{n})\mathcal{A}(t_{\partial\Omega}u_+ - t_{\partial\Omega}u_-) \end{pmatrix}.$$

The right-hand side is in $H^{1/2}(\partial\omega)^8$ and the matrix in the left-hand side is invertible in $H^{1/2}(\partial\Omega)^8$ as long as $\lambda \neq \pm 2$. Thus $t_{\partial\Omega}u_\pm \in H^{1/2}(\partial\Omega)^4$ and $\mathrm{dom}\big(\mathcal{H}_\lambda(m)^*\big) \subset \mathrm{dom}\big(\mathcal{H}_\lambda(m)\big)$. The reciprocal inclusion is similar.

Essential self-adjointness when $\lambda = \pm 2$

Proposition.– $\lambda^2 = 4$.

$$\mathrm{dom}\,(\mathcal{H}_\lambda(m)^*) = \Big\{(u_+, u_-) \in H(\alpha, \Omega_+) \times H(\alpha, \Omega_-) : (u_+, u_-) \text{ satisfies } (*) \text{ in } H^{-1/2}(\partial\Omega)^4\}.$$

Proposition.– Let $\lambda^2 = 4$. The following holds:

$$\overline{\mathcal{H}_\lambda(m)} = \mathcal{H}_\lambda^*(m).$$

In particular, $\overline{\mathcal{H}_\lambda(m)}$ is self-adjoint.

For $u \in \mathrm{dom}\,(\mathcal{H}_\lambda(m)^*)$, Transmission condition reads

$$(***) \qquad t_{\partial\Omega} u_+ = i\varepsilon(\alpha \cdot \mathbf{n}) t_{\partial\Omega} u_-, \qquad \varepsilon = \pm 1$$

as an equality in $H^{-1/2}(\partial\Omega)^4$.

For $u=(u_+,u_-)\in \operatorname{dom}(\mathcal{H}_\lambda(m)^*)$, if $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions $\mathcal{C}^\infty(\partial\Omega)^4$ that converges to $t_{\partial\Omega}u_-$ in the $\|\cdot\|_{H^{-1/2}(\partial\Omega)^4}$-norm, we introduce:

$$(****)\quad \begin{cases} u_{n,-} &= u_- + i\phi_-\Big((\alpha\cdot\mathbf{n})(t_{\partial\Omega}u_- - f_n)\Big), \\ u_{n,+} &= u_+ - \varepsilon\phi_+(f_n - t_{\partial\Omega}u_-) + \varepsilon E_+\Big(\mathcal{A}\big((\alpha\cdot\mathbf{n})(f_n - t_{\partial\Omega}u_-)\big)\Big), \end{cases}$$

Lemma.– Let $u=(u_+,u_-)\in \operatorname{dom}(\mathcal{H}_\lambda(m)^*)$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $\mathcal{C}^\infty(\partial\Omega)^4$ that converges to $t_{\partial\Omega}u_-$ in the $\|\cdot\|_{H^{-1/2}(\partial\Omega)^4}$-norm. If $u_n=(u_{n,-},u_{n,+})$ is the sequence defined in $(****)$ then:

(i) $u_n\in H^1(\Omega_+)^4\times H^1(\Omega_-)^4$,

(ii) $(u_{n,+},u_{n,-})$ satisfies Transmission condition $(***)$ in $H^{1/2}(\partial\Omega)^4$,

(iii) u_n converges to u in the $\|\cdot\|_{H(\alpha,\Omega_+)\times H(\alpha,\Omega_-)}$-norm.

$$\mathrm{dom}\,(\mathcal{H}_\lambda(m)) \subsetneq \mathrm{dom}\,(\overline{\mathcal{H}}_\lambda(m)):$$

Let $0 \neq f \in H^{-1/2}(\partial\Omega)^4$ such that $f \notin H^{1/2}(\partial\Omega)^4$. Either $\mathcal{C}_+(f)$ or $\mathcal{C}_-(f)$ does not belong to $H^{1/2}(\partial\Omega)$. Assume $\mathcal{C}_-(f) \notin H^{1/2}(\partial\Omega)^4$, we set $g = \mathcal{C}_-(f)$. We consider the function

$$u = (u_+, u_-) = \left(\varepsilon\phi_+(g) - \varepsilon E_+\Big(\mathcal{A}\big((\alpha\cdot\mathbf{n})g\big)\Big), \phi_-\big((i\alpha\cdot\mathbf{n})g\big)\right), \qquad (\varepsilon = \pm 1),$$

By definition, $u \in H(\alpha, \Omega_+) \times H(\alpha, \Omega_-)$ and we have:

$$\begin{aligned} i\varepsilon(\alpha\cdot\mathbf{n})t_{\partial\Omega}u_- &= -i\varepsilon(\alpha\cdot\mathbf{n})\mathcal{C}_-(g) \\ &= -\varepsilon i\mathcal{C}_+\big((\alpha\cdot\mathbf{n})g\big) - \varepsilon\mathcal{A}\big((\alpha\cdot\mathbf{n})g\big) \\ &= t_{\partial\Omega}u_+. \end{aligned}$$

Hence u satisfies Transmission condition which gives $u \in \mathrm{dom}\,\big(\overline{\mathcal{H}_\lambda(m)}\big)$. However, $u \notin \mathrm{dom}\,(\mathcal{H}_\lambda(m))$, otherwise $t_{\partial\Omega}u_- \in H^{1/2}(\partial\Omega)^4$ which is not possible because $t_{\partial\Omega}u_- = -g = -\mathcal{C}_-(f) \notin H^{1/2}(\partial\Omega)^4$.

THANK YOU FOR YOUR ATTENTION